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Linear Relationships Between Lengths Associated with a Heptagonal Triangle

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Abstract. A heptagonal triangle is a triangle with angles $\pi/7$, $2\pi/7$, and $4\pi/7$. We use a computer to discover and prove linear relationships between points associated with a heptagonal triangle. For example, if S denotes the first isodynamic point of $\triangle ABC$, then $AS = BS + CS$ in a heptagonal triangle..

Keywords. triangle geometry, Mathematica, computer-discovered mathematics.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

A heptagonal triangle is a triangle with angles $A = \pi/7$, $B = 2\pi/7$, and $C = 4\pi/7$. The following interesting result was found by Rachid Iksi [6].

Theorem 1. *Let $\triangle ABC$ be a heptagonal triangle with largest angle C. Let I, O, and N be the incenter, circumcenter, and nine-point center of $\triangle ABC$ (Figure 1). Then $OI = 2CN$.*

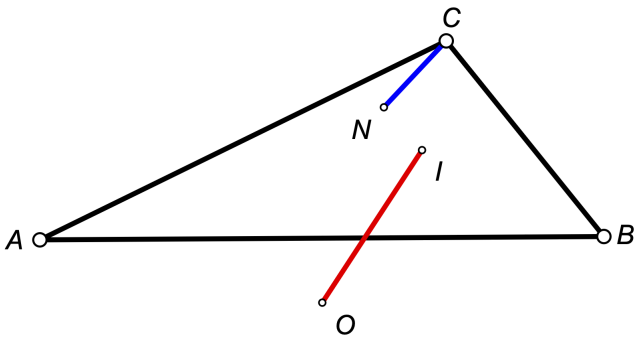

Figure 1. red length = 2×blue length

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This suggests we look for other linear relationships among distances between points associated with a heptagonal triangle. We will always label the heptagonal triangle as ABC, with $AB = c$, $BC = a$, and $CA = b$, with $a < b < c$ as shown in Figure 2.

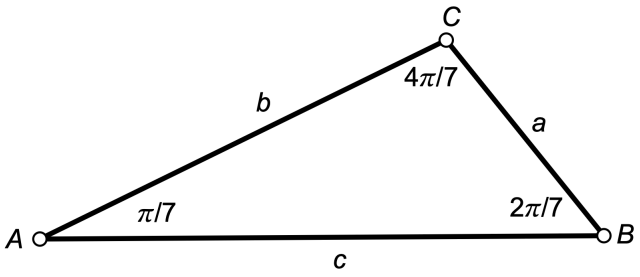

Figure 2. heptagonal triangle

From the Law of Sines, we see that the ratio of the sides is given by

$$a : b : c = \sin\frac{\pi}{7} : \sin\frac{2\pi}{7} : \sin\frac{4\pi}{7}.$$

We use Mathematica to both discover and prove a number of linear relationships similar to Theorem 1 involving distances between the points A, B, and C and common triangle centers. We let X_n denote the n-th named triangle center in the Encyclopedia of Triangle Centers [3]. Our study only includes the centers X_n with $1 \le n \le 20$. We omit any results that are true for all triangles.

2. The Results

The following results do not involve A, B, or C.

Theorem 2. *Let $\triangle ABC$ be a heptagonal triangle. Then*

$$\begin{aligned}
3X_2X_6 &= 4X_5X_6\\
6X_2X_6 &= X_6X_{20}\\
X_3X_6 &= X_4X_6\\
X_3X_{13} &= X_3X_{14}\\
X_4X_{15} &= X_4X_{16}\\
8X_5X_6 &= X_6X_{20}\\
2X_5X_{13} &= X_{13}X_{15}\\
2X_5X_{14} &= X_{14}X_{16}.
\end{aligned}$$

Note: Linear relationships are omitted for points that are collinear in a general triangle. Such points are listed in Appendix A.

The following results involve only one center.

Theorem 3. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$. Then*

$$AX_{15} = BX_{15} + CX_{15}$$
$$AX_{16} = BX_{16} + CX_{16}.$$

Figure 3 illustrates the first result from Theorem 3.

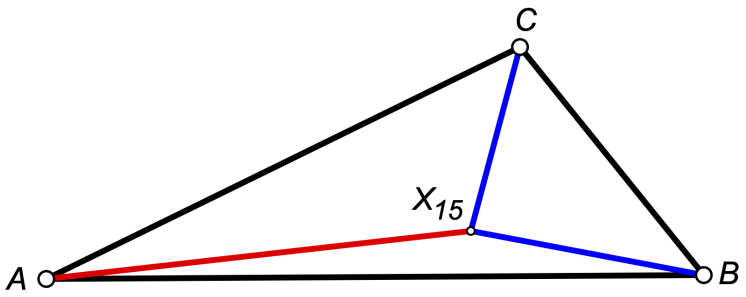

Figure 3. red length = sum of blue lengths

The following results involve one center and AB, BC, AC.

Theorem 4. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$. Then*

$$BX_1 = AC - AB$$
$$CX_1 = AB - AC$$
$$2CX_{10} = BC + AC - AB$$
$$2BX_{11} = BC$$
$$2AX_{12} = AB + AC - BC$$
$$2CX_{12} = AC - BC.$$

Figure 4 illustrates the 4th result from Theorem 4. Note that the two circles in this figure are the incircle and the nine-point circle which touch at X_{11}, the Feuerbach point of $\triangle ABC$. The green dots are the midpoints of the sides. If M is the midpoint of BC, it is interesting to note that $BM = BX_{11}$.

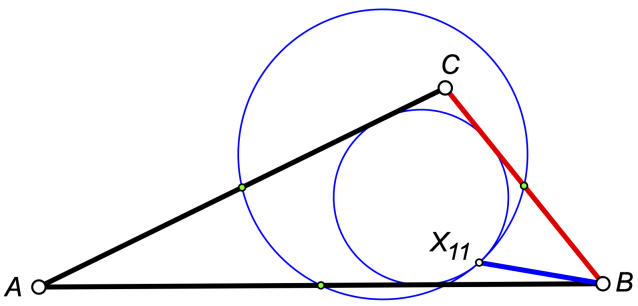

Figure 4. red length = 2×blue length

The following results involve two centers.

Theorem 5. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$. Then*

$$\begin{aligned} AX_1 + CX_4 &= AX_4 \\ BX_1 &= 2CX_{12} \\ AX_5 - BX_5 + CX_5 &= 3X_2X_5 \\ BX_2 &= 2X_2X_{11} \\ AX_5 - BX_5 + CX_5 &= X_4X_5 \\ 2AX_{11} &= AX_4 + BX_4 \\ 2AX_4 - 2BX_4 + CX_4 &= 2X_4X_{12} \\ CX_5 &= X_5X_{10} \\ 3AX_5 - 3BX_5 + 3CX_5 &= X_5X_{20} \\ CX_{10} + 2CX_{12} &= AX_{12} \\ AX_{13} &= CX_{13} + AX_{15} \\ AX_{13} + CX_{15} &= BX_{13} + AX_{15}. \end{aligned}$$

Figure 5 illustrates one of the results from Theorem 5.

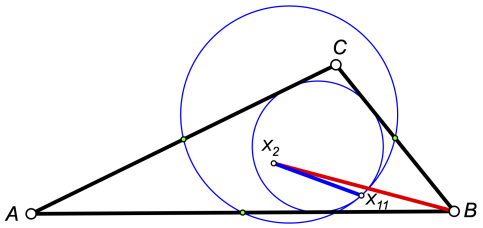

FIGURE 5. red length = 2×blue length

The following results involve two centers and AB, BC, AC.

Theorem 6. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$. Then*

$$\begin{aligned} 2CX_{10} - CX_1 &= BC \\ 7X_3X_6 &= 2AB + 2AC - 2BC \\ 7X_4X_6 &= 2AB + 2AC - 2BC \\ 2CX_{10} + 2CX_{12} &= AB \\ BX_2 + 2BX_{11} - 2X_2X_{11} &= BC. \end{aligned}$$

Figure 6 illustrates the first result from Theorem 6.

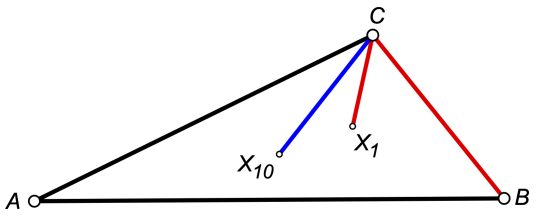

FIGURE 6. sum of red lengths = 2×blue length

The following results involve three centers and vertex A.

Theorem 7. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$. Then*

$$\begin{aligned}
AX_1 + 2X_3X_{11} &= 2AX_3 + 2X_1X_{11}\\
3AX_1 + 6AX_{11} &= 7AX_4 + 2X_1X_{11}\\
2AX_1 + AX_4 + 2X_1X_{12} &= 6X_4X_{12}\\
4AX_3 &= X_3X_8 + 2AX_4\\
2AX_{11} &= AX_4 + 2X_3X_{11}\\
AX_3 &= X_4X_{15}\\
AX_3 &= X_4X_{16}\\
4AX_{12} &= 7X_3X_6\\
4AX_3 + 4X_3X_{11} &= 4AX_{11} + X_3X_8\\
AX_3 &= X_3X_{11} + X_{11}X_{12}\\
AX_4 &= 2X_4X_{12} + 2X_5X_{12}\\
4AX_{12} &= 7X_4X_6\\
2AX_{11} + 4X_4X_{12} &= 3AX_4 + 2X_{11}X_{12}.
\end{aligned}$$

Figure 7 illustrates the 4th result from Theorem 7.

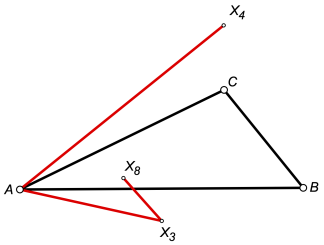

Figure 7. $4AX_3 = X_3X_8 + 2AX_4$

The following results involve three centers and vertex B.

Theorem 8. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$. Then*

$$\begin{aligned}
BX_4 &= 2X_3X_{11}\\
BX_3 &= X_3X_{11} + X_{11}X_{12}\\
BX_{20} &= 2X_4X_{11}.
\end{aligned}$$

Figure 8 illustrates the first result from Theorem 8.

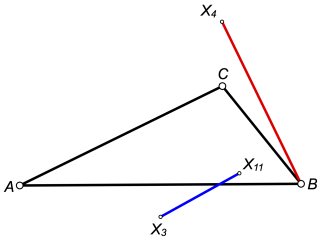

Figure 8. red length = 2×blue length

The following results involve three centers and vertex C.

Theorem 9. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$. Then*

$$\begin{aligned} 2CX_5 &= X_1X_3 \\ 2CX_3 &= CX_4 + 2X_4X_{12} \\ CX_3 &= X_3X_{11} + X_{11}X_{12} \\ 4CX_5 &= X_4X_8 \\ CX_4 + 2X_4X_{12} &= 2X_4X_{15} \\ CX_4 + 2X_4X_{12} &= 2X_4X_{16} \\ CX_{11} &= 2X_{10}X_{12}. \end{aligned}$$

Figure 9 illustrates the 4th result from Theorem 9.

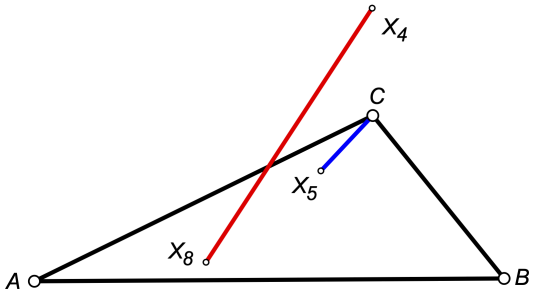

FIGURE 9. red length = 4×blue length

3. The Proofs

The proofs of these results are straightforward using barycentric coordinates. Set up a barycentric coordinate system with $\triangle ABC$ as the reference triangle, so that $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, and $C = (0 : 0 : 1)$.

The barycentric coordinates for the various triangle centers can be found in [3]. These can be expressed in terms of a, b, and c, the lengths of the sides of the reference triangle. The distance between two points in barycentric coordinates can be found using formula 9 from [2].

The theorems involving distances between points whose barycentric coordinates are known can thus be proven by using the distance formula to express all distances in terms of a, b, and c. The proposed theorem then becomes a proposed algebraic identity in a, b, and c. We then let $a = \sin\frac{\pi}{7}$, $b = \sin\frac{2\pi}{7}$, and $c = \sin\frac{4\pi}{7}$ and verify that the resulting formula is true using algebraic and trigonometric simplification with the Mathematica `FullSimplify` command.

As an example, we will give the details of the proof of Theorem 1.

Theorem 10. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$ (Figure 10). Then $X_1X_3 = 2CX_5$.*

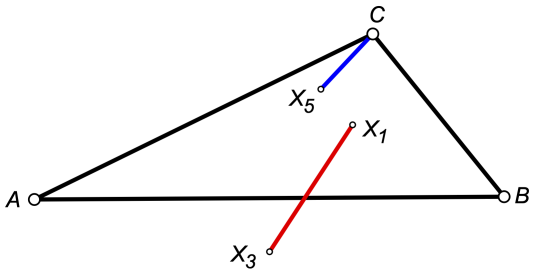

FIGURE 10. red length = 2×blue length

Proof. First we find the barycentric coordinates for each point involved in the theorem. We find

$$\begin{aligned}
C &= (0:0:1)\\
X_1 &= (a:b:c)\\
X_3 &= \left(a^2\left(a^2-b^2-c^2\right) : b^2\left(-a^2+b^2-c^2\right) : c^2\left(-a^2-b^2+c^2\right)\right)\\
X_5 &= \Big(-a^2b^2-a^2c^2+b^4-2b^2c^2+c^4 : a^4-a^2b^2-2a^2c^2-b^2c^2+c^4 :\\
&\qquad a^4-2a^2b^2-a^2c^2+b^4-b^2c^2\Big)
\end{aligned}$$

Then we find the distances between the points.

$$CX_5 = \frac{1}{2}\sqrt{\frac{a^6-a^4b^2-3a^4c^2-a^2b^4-3a^2b^2c^2+3a^2c^4+b^6-3b^4c^2+3b^2c^4-c^6}{(a-b-c)(a+b-c)(a-b+c)(a+b+c)}}$$

$$X_1X_3 = \sqrt{\frac{abc\left(a^3-a^2b-a^2c-ab^2+3abc-ac^2+b^3-b^2c-bc^2+c^3\right)}{(b+c-a)(a+b-c)(a-b+c)(a+b+c)}}$$

Then we apply the substitutions

$$a \to \sin\left(\frac{\pi}{7}\right), b \to \sin\left(\frac{2\pi}{7}\right), c \to \sin\left(\frac{4\pi}{7}\right)$$

and simplify, to get

$$CX_5 = \frac{1}{4}\sqrt{3+2\sin\left(\frac{\pi}{14}\right)-2\sin\left(\frac{3\pi}{14}\right)-2\cos\left(\frac{\pi}{7}\right)}$$

$$X_1X_3 = \frac{1}{2}\sqrt{3+2\sin\left(\frac{\pi}{14}\right)-2\sin\left(\frac{3\pi}{14}\right)-2\cos\left(\frac{\pi}{7}\right)}$$

Finally, simplifying $X_1X_3 - 2CX_5$ gives

$$X_1X_3 - 2CX_5 = 0$$

which concludes the proof. □

4. The Discovery Process

Once an identity involving the distances between points is known, the proof is relatively straightforward using barycentric coordinates. The interesting issue is how to discover such results.

Suppose we want to look for a linear relationship between some or all of a set of numerical distances.

Mathematica has a function called `FindIntegerNullVector` that finds linear relationships between a set of numbers. According to [8], it uses Storjohann's variant of the Lenstra–Lenstra–Lovasz lattice reduction algorithm and is reasonably efficient.

We then use `FindIntegerNullVector` to find a linear relationship between the numerical values.

Suppose we want to find a linear relationship between the distances from the vertices to the first four centers, X_1, X_2, X_3, and X_4. Let `UXn` denote the Mathematica variable who value is the distance from U to X_n after applying the substitutions $a = \sin(\pi/7)$, $b = \sin(2\pi/7)$, and $c = \sin(4\pi/7)$. We then issue the following Mathematica command.

```
FindIntegerNullVector[{AX1,AX2,AX3,AX4,
BX1,BX2,BX3,BX4,CX1,CX2,CX3,CX4}]
```

Mathematica responds with $\{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0\}$. The numbers represent the coefficients of the linear relationship found. In this case, we have found the relationship `1AX3-1CX3=0` or $AX_3 = CX_3$. But this is a well-known result since in any triangle, the circumcenter X_3 is equidistant from the vertices. We therefore remove `BX3` and `CX3` from the list and issue the following Mathematica command.

```
FindIntegerNullVector[{AX1,AX2,AX3,AX4,BX1,BX2,BX4,CX1,CX2,CX4}]
```

This time, Mathematica responds with $\{1, 0, -1, 0, 0, 0, -1, 0, 0, 2\}$ which tells us that it has found the linear relationship `1AX1-1AX3-1BX4+2CX4=0` or

$$AX_1 + 2CX_4 = AX_3 + BX_4.$$

We have thus discovered a new result. This result is stated in Theorem 11.

Theorem 11. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$ (Figure 11). Then $AX_1 + 2CX_4 = AX_3 + BX_4$.*

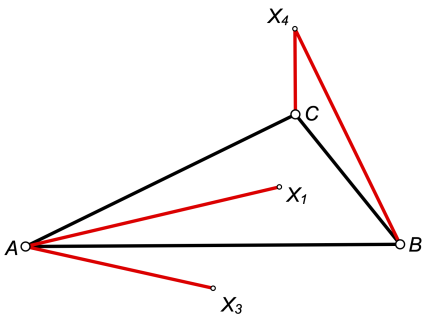

Figure 11. $AX_1 + 2CX_4 = AX_3 + BX_4$

5. Opportunities for Future Research

There are many avenues for future investigation.

5.1. See if these results are exhaustive.

The results found in this paper do not include all possible linear relationships with integer coefficients among the points $A, B, C, X_1, X_2, \ldots, X_{20}$. Find additional relationships that we have missed.

The reason that we did not find all possible relationships is because

(1) We did not investigate all possible combinations of distances between these points.
(2) We constrained our calls to `FindIntegerNullVector` to run for no longer than 60 seconds.
(3) The routine `FindIntegerNullVector` only reports the first linear relationship found amongst its arguments.

5.2. Use more centers.

Iksi [5] found the following result.

Theorem 12. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$ (Figure 12). Then*

$$X_3X_5 + X_5X_{13} = X_5X_{14}.$$

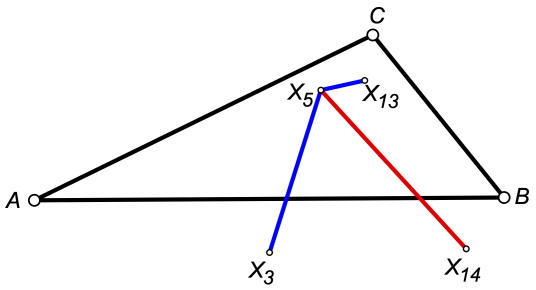

Figure 12. red length = sum of blue lengths

Our study only involved at most three triangle centers, but this result uses four triangle centers (X_3, X_5, X_{13}, and X_{14}). More results can be found by examining configurations containing four or more triangle centers.

5.3. Use centers X_n with $n > 20$.

Our study only involved triangle centers X_n with $1 \leq n \leq 20$. More results can be found by examining configurations containing triangle centers of the form X_n with $n > 20$.

For example, the following result was found by computer.

Theorem 13. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$ (Figure 13). Let X_{99} be the Steiner point of $\triangle ABC$. Then $AX_{99} = CX_4$, $BX_{99} = AX_4$, and $CX_{99} = BX_4$.*

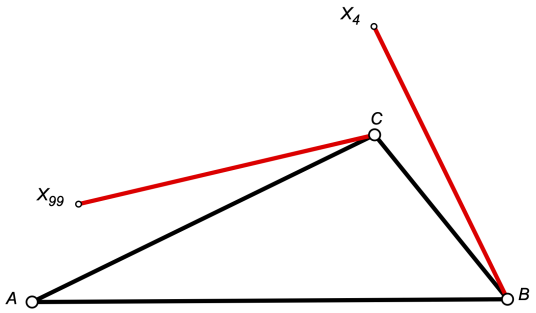

FIGURE 13. red lengths are equal

5.4. **Allow radicals for coefficients.**

Iksi [4] found the following result.

Theorem 14. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$ (Figure 14). Then*

$$X_1X_3 = \sqrt{2} \cdot CX_4.$$

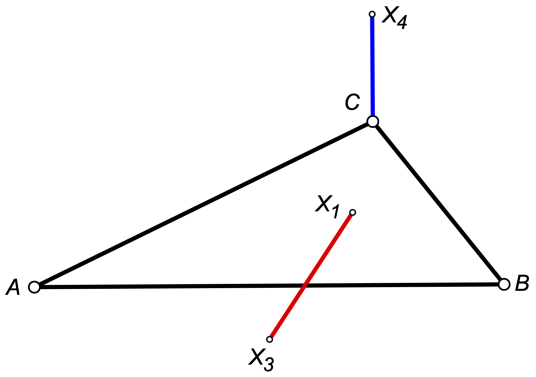

FIGURE 14. red length = $\sqrt{2}\times$blue length

Our study only looked for linear relationships with integer coefficients. More results can be found by allowing the coefficients of the linear relationships to include expressions involving radicals. Although `FindIntegerNullVector` only finds linear relationships with integer coefficients, one can find linear relationships with coefficients in some other field, for example, $\mathbb{Z}[\sqrt{p}]$, by including arguments of the form `XnXm`$\sqrt{p}$ in the call to `FindIntegerNullVector`. Using this technique, the following result was found.

Theorem 15. *Let $\triangle ABC$ be a heptagonal triangle (Figure 15). Then*

$$X_2X_4 = \sqrt{7} \cdot X_2X_6.$$

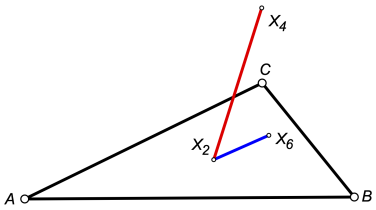

FIGURE 15. red length = $\sqrt{7}\times$blue length

Bamval [1] found the following simple result.

Theorem 16. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$ (Figure 16). Then*

$$AB + AC - BC = AX_3\sqrt{7}.$$

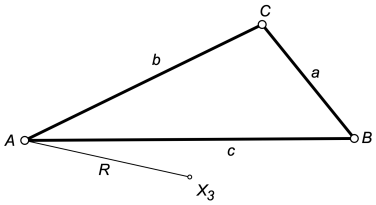

FIGURE 16. $b + c - a = R\sqrt{7}$

The following curious result involving $\sqrt{21}$ was also found by computer.

Theorem 17. *Let $\triangle ABC$ be a heptagonal triangle (Figure 17). Then*

$$X_3X_5 = \left(\frac{3+\sqrt{21}}{2}\right) \cdot X_5X_{13}.$$

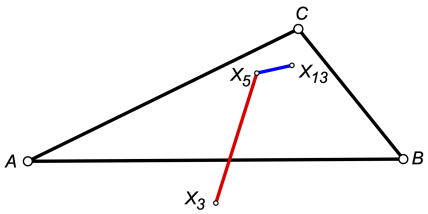

FIGURE 17. red length = $\frac{1}{2}\left(3+\sqrt{21}\right) \times$ blue length

5.5. **Allow powers of distances.**

Instead of finding linear relationships between the various distances, we could study linear relationships between squares of distances, or cubes of distances, etc. For example, the following result was found by computer.

Theorem 18. *Let $\triangle ABC$ be a heptagonal triangle with $a < b < c$ (Figure 18). Then $4(AX_3)^3 + (BX_4)^3 = (AX_4)^3 + (CX_4)^3$.*

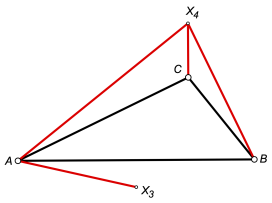

FIGURE 18. $4(AX_3)^3 + (BX_4)^3 = (AX_4)^3 + (CX_4)^3$

5.6. **Allow other shape triangles.**

Instead of looking at triangle centers in a heptagonal triangle, we can look at centers in other shape triangles. We could study right triangles, triangles whose sides form an arithmetic progression, triangles whose sides form a geometric progression, triangles in which one angle is twice another angle, triangles with one angle of 60°, etc. For example, the following result was found by computer.

Theorem 19. *Let $\triangle ABC$ be a triangle in which $\angle C = 2\angle B$ (Figure 19). Then*

$$AC + CX_1 = AB.$$

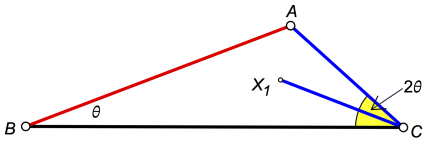

FIGURE 19. red length = sum of blue lengths

5.7. **Allow other points.**

Look for linear relationships involving points that are not triangle centers. For example, the Brocard points of a triangle are not triangle centers.

The following result was found by computer.

Theorem 20. *Let $\triangle ABC$ be a heptagonal triangle (Figure 20). Let Ω_2 be the second Brocard point of $\triangle ABC$, that is, the point whose trilinear coordinates are $(b/c : c/a : a/b)$. Then $X_2X_3 = X_2\Omega_2$.*

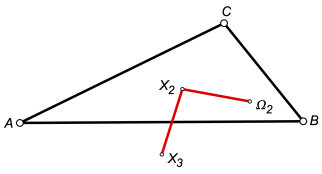

FIGURE 20. red lengths are equal

5.8. **Look for equilateral triangles.**

Look for equilateral triangles among the triangle centers associated with a heptagonal triangle. For example, the following result is Theorem 22 in [9].

Theorem 21. *Let $\triangle ABC$ be a heptagonal triangle (Figure 21). Then $\triangle X_3X_{13}X_{14}$ is an equilateral triangle.*

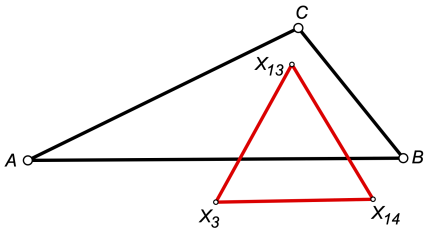

Figure 21. red figure is an equilateral triangle

5.9. **Look for more general results.**

The results in this paper are all about heptagonal triangles. See if any of these results apply to a larger class of triangles. For example, the set of heptagonal triangles is a subset of the set of all triangles in which one angle is twice another angle. The set of heptagonal triangles is a subset of the set of all triangles in which one angle is four times another angle. The set of heptagonal triangles is a subset of the set of all triangles with sides of lengths a, b, and c that satisfy $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$. For each result in this paper, see if the result applies to some larger class of triangles.

For example, Theorem 3 says that in a heptagonal triangle, $AX_{15} = BX_{15}+CX_{15}$. This turns out to be true for a wider class of triangles given by the following theorem.

Theorem 22. *Let $\triangle ABC$ be a triangle with sides of lengths a, b, and c satisfying $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ (Figure 22). Then $AX_{15} = BX_{15} + CX_{15}$.*

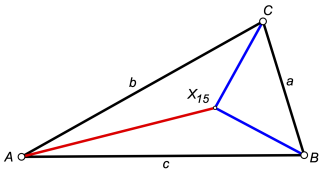

Figure 22. red length = sum of blue lengths

5.10. **Find geometric proofs.**

Can we find purely geometric proofs for any of the results in this paper?

Appendix A. Collinear points

The following sets of points are collinear in any triangle [3].

$\{X_1, X_2, X_8, X_{10}\}$
$\{X_1, X_5, X_{11}, X_{12}\}$
$\{X_1, X_6, X_9\}$
$\{X_1, X_7, X_{20}\}$
$\{X_2, X_3, X_4, X_5, X_{20}\}$
$\{X_2, X_7, X_9\}$
$\{X_2, X_{13}, X_{16}\}$
$\{X_2, X_{14}, X_{15}\}$
$\{X_3, X_6, X_{15}, X_{16}\}$
$\{X_4, X_9, X_{10}, X_{19}\}$
$\{X_4, X_{15}, X_{17}\}$
$\{X_4, X_{16}, X_{18}\}$
$\{X_3, X_{13}, X_{17}\}$
$\{X_3, X_{14}, X_{18}\}$
$\{X_5, X_{13}, X_{18}\}$
$\{X_5, X_{14}, X_{17}\}$
$\{X_6, X_{13}, X_{14}\}$
$\{X_6, X_{17}, X_{18}\}$

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