International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 10, 2025, pp.546-561
Received July 16, 2025. Published on-line October 29, 2025
web: http://www.journal-1.eu/
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Four high precision approximations of a regular heptagon using a marked ruler and compass

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Abstract. The ancient Greek problem of constructing an exact regular heptagon using a straight edge and a compass is known to be impossible since 1837. Several mathematicians have tried elegant ways to achieve a good approximation of the same. This work presents a construction of a regular heptagon using a marked ruler and a compass with the help of elementary trigonometry. Computer programming has been used to find practically feasible triangles of specified side lengths such that one angle of that triangle is very close to $\left(\frac{360}{7}\right)^\circ$. The central angle of the approximate regular heptagon is correct up to 8 places of decimals. A total of four constructions based on the same underlying pattern have been presented in this paper.

Keywords. heptagon, approximate, compass, ruler, trigonometry, computer-discovered

Mathematics Subject Classification (2010). 51-04, 65-04, 97G60.

1. Introduction

The ancient Greek mathematicians proposed the problem of constructing any regular polygon using a straight edge and a compass. Gauss–Wantzel Theorem shows that a regular polygon of n sides can be constructed by a compass and a straight edge if and only if n is a product of any whole number power of 2 and distinct Fermat primes. The number 7 is a prime number, but not a Fermat prime, thus making it impossible to construct an exact regular heptagon using a straight edge and compass.

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Also, it is not possible to construct an exact angle of $\left(\frac{360}{7}\right)^\circ$ using a protractor or any other present day geometrical instruments.

Numerous attempts have been made by several mathematicians over the years to get a good approximate construction of a regular heptagon.

1.1. **Survey of existing work.** Sir Hamilton [1] proposed a construction of the regular heptagon with an error of just $0.04''$. A note by Williamson [2] suggests using a central angle of $tan^{-1}\left(\frac{5}{4}\right)$ as the central angle of a heptagon. This construction is very simple, but has an error of $5'$ which is a bit on the higher side.

The construction by Johnson [3] involves a tricky manoeuvre of placing a ruler in a specific way, which may be prone to errors. Wunsche [4] proposes yet another unconventional method using a conceptual instrument named rhombic bicompasses along with a ruler.

Lastly, a very recent paper by Pfeifer [5] gives a very high precision approximation of a regular heptagon with a very low absolute percentage error of around $(2.47268 \times 10^{-9})\,\%$ in the ***sine*** *of the central angle*. This translates to an absolute percentage error of nearly $(2.73477 \times 10^{-9})\,\%$ in the ***measurement*** *of the central angle* But, the lengths of line segments used in this construction *need to be* ***scaled*** *carefully* to enable construction on a standard A2 sheet of paper, which is typically used for engineering drawing.

1.2. **Outline of this work. Three different constructions** of a regular heptagon have been proposed in this work *using a marked ruler and compass*. All of these constructions are feasible on a standard $A2$ sheet of paper, which measures $(42\ cm \times 59.4\ cm)$, making them suitable for practical use by engineers and architects. The error is of the order of $(10^{-8})^\circ$.

A fourth construction is achieved by *combining two of these constructions* and the absolute percentage error in the ***measurement*** *of the central angle* in this construction is $\mathbf{(9.58285 \times 10^{-10})\,\%}$, which is **nearly $\frac{1}{3}$ of the error** in the construction presented by Pfeifer[5]. This fourth construction is also achievable on a standard A2 sheet of paper *without any scaling*.

Elementary law of cosines has been used to find a simple planar triangle containing an angle of almost $\left(\frac{360}{7}\right)^\circ$ with high precision. Another interesting aspect of this work is the use of **computer programming** to find suitable measurements of a triangle which results in a close approximation of the aforesaid angle.

2. Geometrical preliminaries

First of all, a few elementary geometrical details are discussed, which are relevant to these constructions. After that, all constructions are described in detail, leveraging these concepts.

2.1. **Exact construction of a line segment of length $\sqrt{M}$ when M is a natural number.** The key idea of construction comes from **Pythagoras Theorem** and **Lagrange's four-square theorem**. Lagrange's four-square theorem

states that *any positive integer can be expressed as a sum of* ***at most 4 perfect squares***.

- If M is a perfect square, then the construction is trivial using a marked ruler.
- If M can be expressed as a sum of **two perfect squares** in the form $\boldsymbol{M = p^2 + q^2}$, then it is sufficient to draw a right triangle with the perpendicular sides of lengths p units and q units.

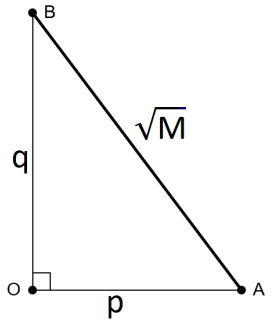

FIGURE 1. $\boldsymbol{M = p^2 + q^2}$

The length of hypotenuse AB is $\boldsymbol{\sqrt{M}}$.

- If M can be expressed as a sum of **three perfect squares** in the form $\boldsymbol{M = p^2 + q^2 + r^2}$, then it is required to construct two right triangles as shown below.

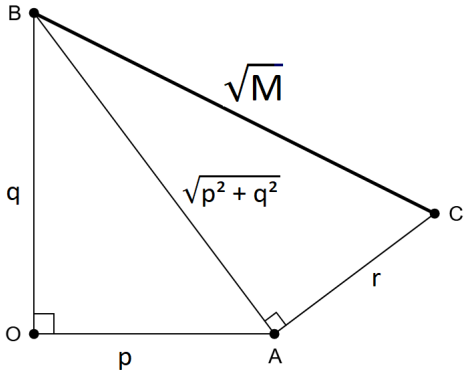

FIGURE 2. $\boldsymbol{M = p^2 + q^2 + r^2}$

Here, $OA = p$ units, $OB = q$ units and $AC = r$ units.
$OB \perp OA$ and $AC \perp AB$.
By Pythagoras Theorem, $AB^2 = OA^2 + OB^2 = p^2 + q^2$
and, $BC = \sqrt{AB^2 + AC^2} = \sqrt{p^2 + q^2 + r^2} = \sqrt{M}$

- If M can be expressed as a sum of **four perfect squares** in the form $\boldsymbol{M = p^2 + q^2 + r^2 + s^2}$, then it is required to construct three right triangles as shown below.

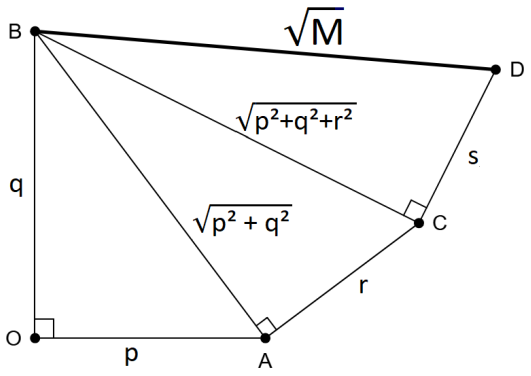

FIGURE 3. $\boldsymbol{M = p^2 + q^2 + r^2 + s^2}$

Here, $OA = p$ units, $OB = q$ units, $AC = r$ units and $CD = s$ units.
$OB \perp OA$, $AC \perp AB$ and $CD \perp BC$.
By Pythagoras Theorem, $AB^2 = OA^2 + OB^2 = p^2 + q^2$
$BC^2 = AB^2 + AC^2 = p^2 + q^2 + r^2$
and, $CD = \sqrt{BC^2 + CD^2} = \sqrt{p^2 + q^2 + r^2 + s^2} = \sqrt{M}$

To summarize, in order to construct a line segment of length $\sqrt{M}$ for $M \in Z^+$, express the number M as the sum of **least possible number of** perfect squares, and then perform the appropriate construction shown in **Figure 1**, **Figure 2** or **Figure 3**.

Further, there exist multiple ways of expressing several positive integers as sum of perfect squares. In such cases, the combination of perfect squares needs to be chosen carefully to fit the construction seamlessly on a standard A2 sheet of paper.

2.2. **Constructing a regular polygon with specified circumradius or with specified length of side.** There are two common flavours of constructing a regular polygon:

- A regular polygon with a specified circumradius
- A regular polygon with a specified length of its sides

A brief outline of construction of both of these flavours is presented below.

2.2.1. *Constructing a regular heptagon with a specified circumradius.* If a circumcircle of a regular heptagon is constructed, then each side of that polygon makes an angle $\theta^\circ = \left(\frac{360}{7}\right)^\circ$ at the center of the circumcircle.
Once the **central angle** θ° has been obtained with good accuracy, and the radius of circumcircle is known, the following steps illustrate the construction of the corresponding polygon.

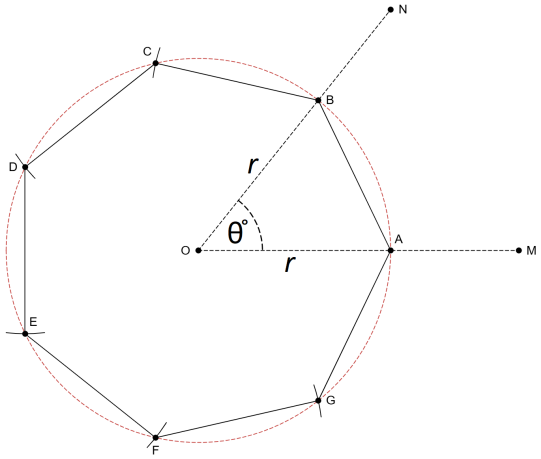

FIGURE 4. A regular heptagon with central angle θ° and circumradius r units

The steps of this construction are listed below.

(1) Construct the desired central angle $\angle MON$ measuring $\theta^\circ \approx \left(\frac{360}{7}\right)^\circ$. *The construction of this approximate angle is discussed in detail in* ***Section 3****.*
(2) Taking O as center and radius $= r$ (the specified radius of the circumcircle), construct a circle, which intersects OM at A, and ON at B, respectively.
(3) Taking B as center and radius $= AB$, draw an arc intersecting the circle at C.
(4) Taking C as center and radius $= AB$, draw an arc intersecting the circle at D.
(5) Taking D as center and radius $= AB$, draw an arc intersecting the circle at E.
(6) Taking E as center and radius $= AB$, draw an arc intersecting the circle at F.
(7) Taking F as center and radius $= AB$, draw an arc intersecting the circle at G.
(8) Join the line segments AB, BC, CD, DE, EF, FG and GA.

2.2.2. *Constructing a regular polygon with specified length of side.* A few additional steps are needed to construct a regular polygon with side $= x$ *units.*

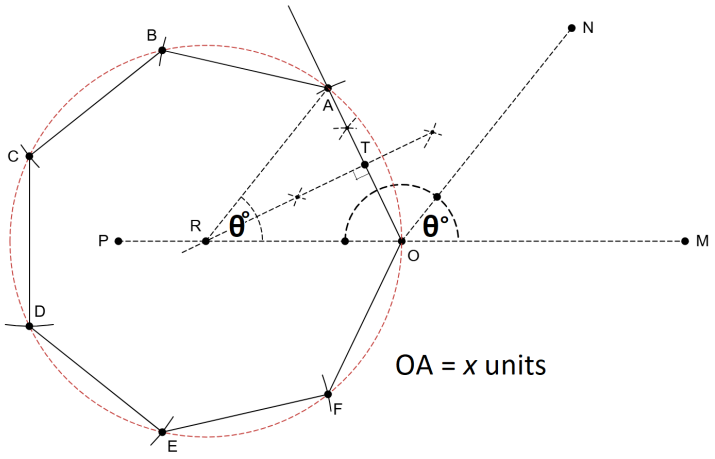

Figure 5. A regular heptagon with central angle θ° and side x units

(1) Construct the desired central angle $\angle MON$ measuring $\theta^\circ \approx \left(\frac{360}{7}\right)^\circ$. *The construction of this approximate angle is discussed in detail in* ***Section 3****.*
(2) Extend MO to an arbitrary point P. Clearly, $\angle NOP = (180^\circ - \theta^\circ)$.
(3) Draw the angle bisector of $\angle NOP$.
(4) Taking O as center and radius = ***OA = x* units**, draw an arc intersecting the angle bisector of $\angle NOP$ at a point A. Clearly, $\angle TOR = \angle AOR = \frac{\angle NOP}{2} = \left(90^\circ - \frac{\theta^\circ}{2}\right)$.
(5) Draw the perpendicular bisector of OA and extend it to intersect OP at a point R. Clearly, $OT = TA$ and $OT \perp RT$. So, $\angle ORT = 180^\circ - (\angle TOR + \angle OTR) = \frac{\theta^\circ}{2}$.
Finally, $\angle ARO = 2 \times \angle ORT = \theta^\circ$
This proves that OA is a side of the regular heptagon with the desired length of x *units*.
(6) Taking R as center and radius = RO, construct a circle. This circle passes through the points O and A.
(7) Taking A as center and radius = OA, draw an arc intersecting the circle at a point B.

From this point onwards, the construction is similar to the one discussed in **Section 2.2.1**.

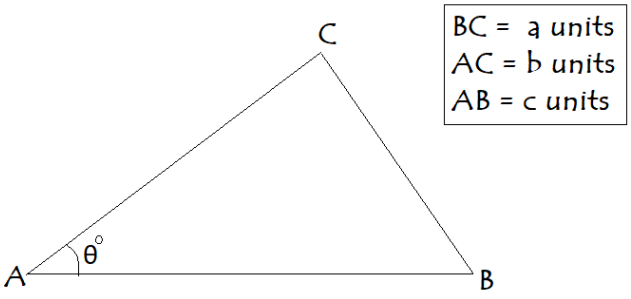

Figure 6. A generic triangle

2.3. **The law of cosines.**
The elementary law of cosines gives the formula

$$\cos\angle BAC = \frac{AC^2 + AB^2 - BC^2}{2 \times AC \times AB}$$

$$\Rightarrow \boxed{\cos\theta^\circ = \frac{b^2 + c^2 - a^2}{2bc}} \tag{1}$$

3. THE FOUR CONSTRUCTIONS OF APPROXIMATE REGULAR HEPTAGON

Computer programming was used to find suitable values of the side lengths (a, b, c) which give very close approximations of the central angle $\boldsymbol{\theta_c^\circ = \left(\frac{360}{7}\right)^\circ}$ in accordance with **Equation 1**. The values are listed below.

Table 1: Data for approximation of regular heptagon

(a, b, c)	$\boldsymbol{\cos\theta^\circ}$ *	$\boldsymbol{\theta^\circ}$ **	$\boldsymbol{\epsilon}$ *** $\boldsymbol{= (\theta^\circ - \theta_c^\circ)}$	$\boldsymbol{\delta}$ **** $\boldsymbol{= \left(\frac{\epsilon \times 100}{\theta_c^\circ}\right)\%}$
$\left(\sqrt{534}, \sqrt{227}, \sqrt{857}\right)$	$\frac{275}{\sqrt{194539}}$	**51.4285714**330°	$+\left(4.39133 \times 10^{-9}\right)^\circ$	$+\left(8.53870 \times 10^{-9}\right)\%$
$\left(\sqrt{665}, \sqrt{908}, \sqrt{857}\right)$	$\frac{275}{\sqrt{194539}}$	**51.4285714**330°	$+\left(4.39133 \times 10^{-9}\right)^\circ$	$+\left(8.53870 \times 10^{-9}\right)\%$
$\left(\sqrt{781}, \sqrt{1023}, \sqrt{1051}\right)$	$\frac{1293}{2\sqrt{1075173}}$	**51.42857142**52°	$-\left(3.40567 \times 10^{-9}\right)^\circ$	$-\left(6.62213 \times 10^{-9}\right)\%$

* $\cos\theta^\circ = \left(\frac{b^2+c^2-a^2}{2bc}\right)$ from **Equation (1)**.
The key idea is to construct a triangle with these measurements as shown in **Figure 6**.

** θ° is **rounded** to 10 places of decimals.
*** ϵ is the error in the approximate central angle θ°.
**** δ is the percentage error in the approximate central angle θ°.

If the **arithmetic mean** of the θ° values is taken for the first (or second) and the third rows, it gives

$$\theta_1^\circ = \frac{\cos^{-1}\left(\frac{275}{\sqrt{194539}}\right) + \cos^{-1}\left(\frac{1293}{2\sqrt{1075173}}\right)}{2} = \mathbf{51.42857142}906426\ldots^\circ$$

The error ϵ for the angle θ_1° is

$$\boldsymbol{\epsilon} = (\theta_1^\circ - \theta_c^\circ) = \mathbf{51.42857142}906426\ldots^\circ - \left(\frac{360}{7}\right)^\circ \approx +\left(4.92832 \times 10^{-10}\right)^\circ$$

and the percentage error is

$$\boldsymbol{\delta} = \left(\frac{\epsilon \times 100}{\theta_c^\circ}\right)\% = \left(\frac{\left(4.92832 \times 10^{-10}\right) \times 100}{\left(\frac{360}{7}\right)}\right)\% \approx \boldsymbol{+\left(9.58285 \times 10^{-10}\right)\%}$$

The percentage error δ in this case is **nearly $\frac{1}{3}$ of the error** in the construction presented by Pfeifer[5]. So, **combining** the *first (or second)* and the *third*

constructions from **Table 1** gives a highly accurate construction of a regular heptagon. This construction is described in **Section 3.4**.

Based on these discussions and the data in **Table 1**, the detailed constructions are presented below.

3.1. **First construction.** Take the data from the *first row* of **Table 1**. The key idea is to construct a triangle as shown in **Figure 6** with

$a = \sqrt{534}\ units = \sqrt{19^2 + 13^2 + 2^2}\ units$ (refer to **Figure 2**)

$b = \sqrt{227}\ units = \sqrt{11^2 + 9^2 + 5^2}\ units$ (refer to **Figure 2**)

$c = \sqrt{857}\ units = \sqrt{29^2 + 4^2}\ units$ (refer to **Figure 1**)

The combination of perfect squares listed above ensures that the construction fits easily within the boundaries of a standard A2 sheet of paper.

Once the desired central angle of $\cos^{-1}\left(\frac{275}{\sqrt{194539}}\right) \approx \left(\frac{360}{7}\right)^\circ$ has been obtained, the desired heptagon can be constructed as shown in **Section 2.2.1** or **Section 2.2.2**. For the sake of simplicity, the construction is shown for a heptagon inscribed in a circle of a specified radius, as discussed in **Section 2.2.1**.

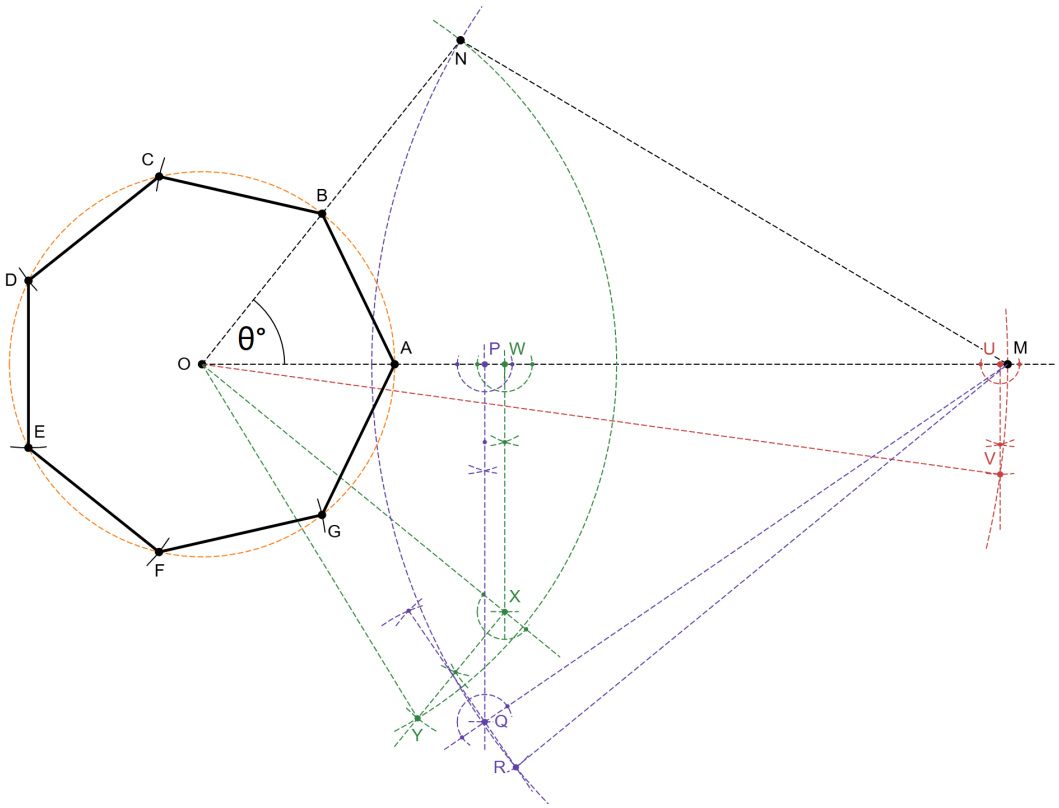

Figure 7. First construction of an approximate regular heptagon

The main steps of this construction are listed below.

(1) Draw a horizontal line segment $OU = 29\,cm$. Draw $UV \perp OU$ such that $UV = 4\,cm$. Join OV.
(2) Taking O as center and radius = OV, draw an arc intersecting *extended* line segment OU at a point M. By Pythagoras Theorem, $\boldsymbol{OM} = OV = \sqrt{OU^2 + UV^2} = \sqrt{29^2 + 4^2} = \boldsymbol{\sqrt{857}\,cm}$
(3) Mark a point W on OU such that $OW = 11\,cm$. Draw $WX \perp OW$ such that $WX = 9\,cm$. Join OX. By Pythagoras Theorem, $OX = \sqrt{OW^2 + WX^2} = \sqrt{11^2 + 9^2} = \sqrt{202}\,cm$.
(4) Draw $XY \perp OX$ such that $XY = 5\,cm$. Join OY. By Pythagoras Theorem, $\boldsymbol{OY} = \sqrt{OX^2 + XY^2} = \sqrt{202 + 5^2} = \boldsymbol{\sqrt{227}\,cm}$.
(5) Mark a point P on OU such that $MP = 19\,cm$. Draw $PQ \perp MP$ such that $PQ = 13\,cm$. Join MQ. By Pythagoras Theorem, $MQ = \sqrt{MP^2 + PQ^2} = \sqrt{19^2 + 13^2} = \sqrt{530}\,cm$.
(6) Draw $QR \perp MQ$ such that $QR = 2\,cm$. Join MR. By Pythagoras Theorem, $\boldsymbol{MR} = \sqrt{MQ^2 + QR^2} = \sqrt{530 + 2^2} = \boldsymbol{\sqrt{534}\,cm}$.
(7) Taking O as center and radius = $\boldsymbol{OY = \sqrt{227}\,cm}$, draw an arc. Taking M as center and radius = $\boldsymbol{MR = \sqrt{534}\,cm}$, draw another arc intersecting the previous arc at a point N. Join ON and MN.
Clearly, $OM = \sqrt{857}\,cm$, $ON = OY = \sqrt{227}\,cm$, *and* $MN = MR = \sqrt{534}\,cm$.

$$\cos\theta^\circ = \cos\angle MON = \frac{ON^2 + OM^2 - MN^2}{2 \times ON \times OM} = \frac{227 + 857 - 534}{2 \times \sqrt{227} \times \sqrt{857}} = \frac{275}{\sqrt{194539}}$$

$$\theta^\circ = \cos^{-1}\left(\frac{275}{\sqrt{194539}}\right) \approx \mathbf{51.428571}4330^\circ \approx \left(\frac{\mathbf{360}}{\mathbf{7}}\right)^\circ$$

(8) As the desired *central angle* of the regular heptagon has been approximated reasonably, so from this point onwards, the steps discussed in **Section 2.2.1** can be applied verbatim for completing the heptagon ***ABCDEFG***.

3.2. **Second construction.** Take the data from the *second row* of **Table 1**. Construct a triangle as shown in **Figure 6** with

$a = \sqrt{665}\,units = \sqrt{25^2 + 6^2 + 2^2}\,units$ (refer to **Figure 2**)

$b = \sqrt{908}\,units = \sqrt{26^2 + 14^2 + 6^2}\,units$ (refer to **Figure 2**)

$c = \sqrt{857}\,units = \sqrt{29^2 + 4^2}\,units$ (refer to **Figure 1**)

For the sake of simplicity, the construction is shown for a heptagon inscribed in a circle of a specified radius, as discussed in **Section 2.2.1**.

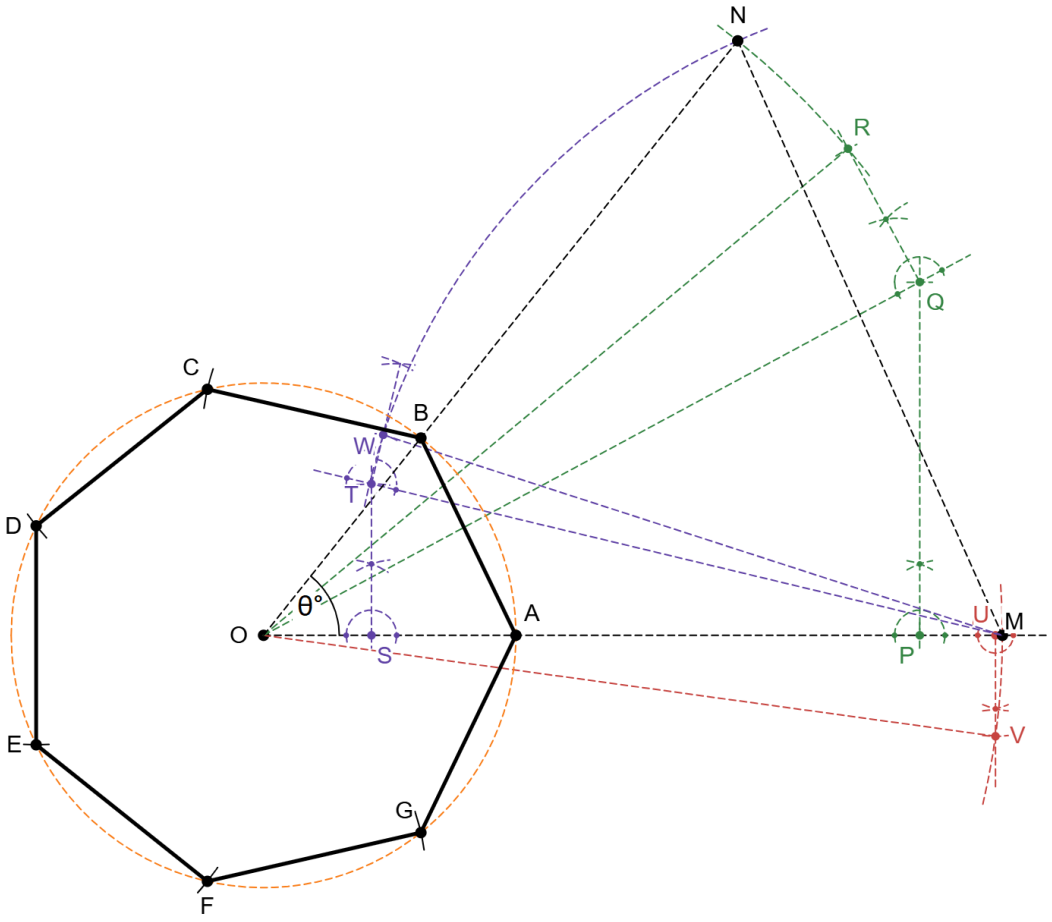

FIGURE 8. Second construction of an approximate regular heptagon

The main steps of this construction are listed below.

(1) Draw a horizontal line segment $OU = 29\,cm$. Draw $UV \perp OU$ such that $UV = 4\,cm$. Join OV.
(2) Taking O as center and radius $= OV$, draw an arc intersecting *extended* line segment OU at a point M. By Pythagoras Theorem, $\boldsymbol{OM} = OV = \sqrt{OU^2 + UV^2} = \sqrt{29^2 + 4^2} = \boldsymbol{\sqrt{857}\,cm}$
(3) Mark a point P on OM such that $OP = 26\,cm$. Draw $PQ \perp OP$ such that $PQ = 14\,cm$. Join OQ. By Pythagoras Theorem, $OQ = \sqrt{OP^2 + PQ^2} = \sqrt{26^2 + 14^2} = \sqrt{872}\,cm$.
(4) Draw $QR \perp OQ$ such that $QR = 6cm$. Join OR. By Pythagoras Theorem, $\boldsymbol{OR} = \sqrt{OQ^2 + QR^2} = \sqrt{872 + 6^2} = \boldsymbol{\sqrt{908}\,cm}$.
(5) Mark a point S on OM such that $MS = 25\,cm$. Draw $ST \perp MS$ such that $ST = 6\,cm$. Join MT. By Pythagoras Theorem, $MT = \sqrt{MS^2 + ST^2} = \sqrt{25^2 + 6^2} = \sqrt{661}\,cm$.
(6) Draw $TW \perp MT$ such that $TW = 2\,cm$. Join MW. By Pythagoras Theorem, $\boldsymbol{MW} = \sqrt{MT^2 + TW^2} = \sqrt{661 + 2^2} = \boldsymbol{\sqrt{665}\,cm}$.
(7) Taking O as center and radius $= \boldsymbol{OR = \sqrt{908}\,cm}$, draw an arc. Taking M as center and radius $= \boldsymbol{MW = \sqrt{665}\,cm}$, draw another arc intersecting the previous arc at a point N. Join ON and MN.

Clearly, $OM = \sqrt{857}\,cm$, $ON = OR = \sqrt{908}\,cm$, *and* $MN = MW = \sqrt{665}\,cm$.

$$\cos\theta^\circ = \cos\angle MON = \frac{ON^2 + OM^2 - MN^2}{2 \times ON \times OM} = \frac{908 + 857 - 665}{2 \times \sqrt{908} \times \sqrt{857}} = \frac{275}{\sqrt{194539}}$$

$$\theta^\circ = \cos^{-1}\left(\frac{275}{\sqrt{194539}}\right) \approx \mathbf{51.4285714}330^\circ \approx \left(\frac{\mathbf{360}}{\mathbf{7}}\right)^\circ$$

It can be seen that this angle is exactly same as the angle obtained in ***Section 3.1****.*

(8) As the desired *central angle* of the regular heptagon has been approximated reasonably, so from this point onwards, the steps discussed in **Section 2.2.1** can be applied verbatim for completing the heptagon ***ABCDEFG***.

3.3. **Third construction.** Take the data from the *third row* of **Table 1**. Construct a triangle as shown in **Figure 6** with

$a = \sqrt{781}\,units = \sqrt{27^2 + 6^2 + 4^2}\,units$ (refer to **Figure 2**)

$b = \sqrt{1023}\,units = \sqrt{31^2 + 7^2 + 3^2 + 2^2}\,units$ (refer to **Figure 3**)

$c = \sqrt{1051}\,units = \sqrt{31^2 + 9^2 + 3^2}\,units$ (refer to **Figure 2**)

For the sake of simplicity, the construction is shown for a heptagon inscribed in a circle of a specified radius, as discussed in **Section 2.2.1**.

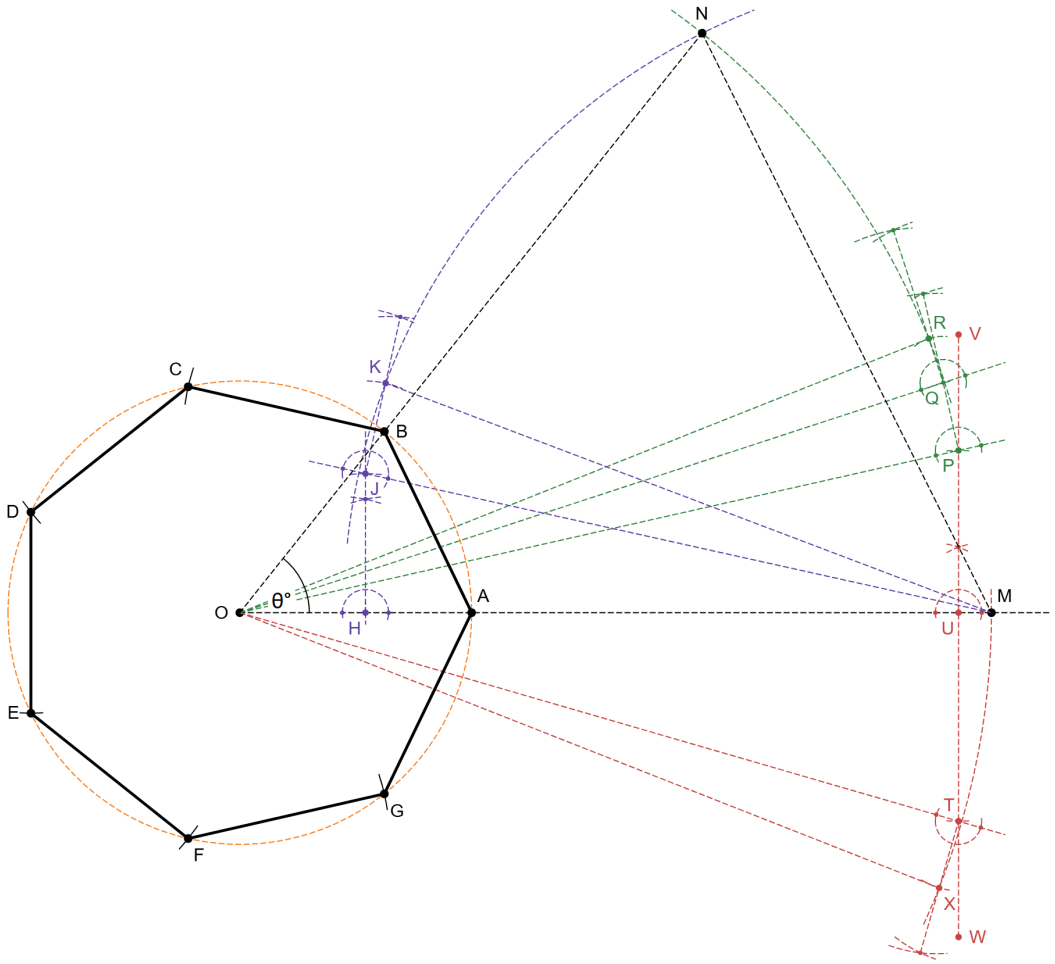

FIGURE 9. Third construction of an approximate regular heptagon

The main steps of this construction are listed below.

(1) Draw a horizontal line segment $OU = 31cm$. Draw a line segment $VUW \perp OU$.
(2) Taking U as center and radius $= 9\,cm$, draw an arc intersecting UW at a point T. Join OT. By Pythagoras Theorem, $OT = \sqrt{OU^2 + UT^2} = \sqrt{31^2 + 9^2} = \sqrt{1042}\ cm$.
(3) Draw $TX \perp OT$ such that $TX = 3\,cm$. Join OX. Taking O as center and radius $= OX$, draw an arc intersecting OU (*extended*) at a point M. By Pythagoras Theorem, $\boldsymbol{OM} = OX = \sqrt{OT^2 + TX^2} = \sqrt{1042 + 3^2} = \boldsymbol{\sqrt{1051}\,cm}$.
(4) Taking U as center and radius $= 7\,cm$, draw an arc intersecting UV at a point P. Join OP. By Pythagoras Theorem, $OP = \sqrt{OU^2 + UP^2} = \sqrt{31^2 + 7^2} = \sqrt{1010}\ cm$.
(5) Draw $PQ \perp OP$ such that $PQ = 3cm$. Join OQ. By Pythagoras Theorem, $OQ = \sqrt{OP^2 + PQ^2} = \sqrt{1010 + 3^2} = \sqrt{1019}\,cm$.
(6) Draw $QR \perp OQ$ such that $QR = 2cm$. Join OR. By Pythagoras Theorem, $\boldsymbol{OR} = \sqrt{OQ^2 + QR^2} = \sqrt{1019 + 2^2} = \boldsymbol{\sqrt{1023}\,cm}$.

(7) Taking M as center and radius = $27\,cm$, draw an arc intersecting OM at a point H. Draw $HJ \perp MH$ such that $HJ = 6\,cm$. Join MJ. By Pythagoras Theorem, $MJ = \sqrt{MH^2 + HJ^2} = \sqrt{27^2 + 6^2} = \sqrt{765}\,cm$.

(8) Draw $JK \perp MJ$ such that $JK = 4\,cm$. Join MK. By Pythagoras Theorem, $\boldsymbol{MK} = \sqrt{MJ^2 + JK^2} = \sqrt{765 + 4^2} = \boldsymbol{\sqrt{781}\,cm}$.

(9) Taking O as center and radius = $\boldsymbol{OR = \sqrt{1023}\,cm}$, draw an arc. Taking M as center and radius = $\boldsymbol{MK = \sqrt{781}\,cm}$, draw another arc intersecting the previous arc at a point N. Join ON and MN.
Clearly, $OM = \sqrt{1051}\,cm$, $ON = OR = \sqrt{1023}\,cm$, *and* $MN = MK = \sqrt{781}\,cm$.

$$\cos\theta^\circ = \cos\angle MON = \frac{ON^2 + OM^2 - MN^2}{2 \times ON \times OM} = \frac{1023 + 1051 - 781}{2 \times \sqrt{1023} \times \sqrt{1051}} = \frac{1293}{2\sqrt{1075173}}$$

$$\theta^\circ = \cos^{-1}\left(\frac{1293}{2\sqrt{1075173}}\right) \approx \mathbf{51.4285714}252^\circ \approx \left(\frac{\mathbf{360}}{\mathbf{7}}\right)^\circ$$

(10) As the desired *central angle* of the regular heptagon has been approximated reasonably, so from this point onwards, the steps discussed in **Section 2.2.1** can be applied verbatim for completing the heptagon $\boldsymbol{ABCDEFG}$.

3.4. **Combining the First and Third constructions.** This is the final construction, which combines the first construction from **Section 3.1** and the third construction from **Section 3.3** to get a highly accurate approximation of a regular heptagon.

The idea is to **add up** the two approximate angles obtained in those two constructions, and then **bisect** the *sum* of those angles.

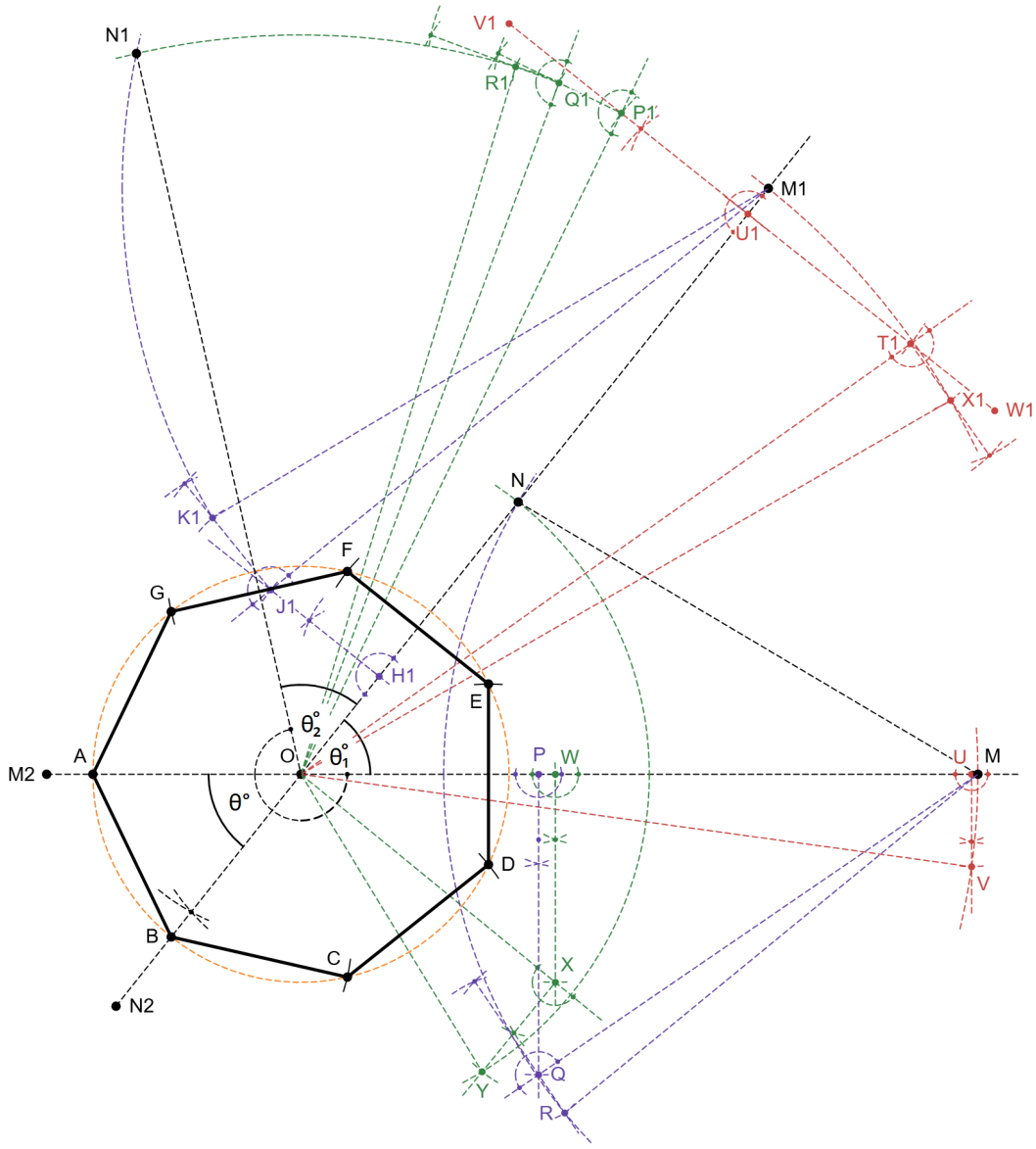

FIGURE 10. Combining the two constructions from Figure 7 and Figure 9

The main steps of this construction are listed below.

(1) Perform the construction of ***∠MON*** according to steps described in **Section 3.1**.

$$\angle MON = \theta_1^\circ = \cos^{-1}\left(\frac{275}{\sqrt{194539}}\right) \approx \mathbf{51.4285714}330^\circ \approx \left(\frac{\mathbf{360}}{\mathbf{7}}\right)^\circ$$

(2) Next, perform the construction according to steps described in **Section 3.3** with the following minor modifications :
   - To avoid naming conflicts, the respective vertices are to be renamed as U_1, V_1, W_1 etc.

- The line segment OU_1 is to be a part of the *(extended)* line segment ON constructed in previous step.
- Construct $\angle M_1ON_1$ according to the steps described in **Section 3.3**.

$$\angle M_1ON_1 = \theta_2^\circ = \cos^{-1}\left(\frac{1293}{2\sqrt{1075173}}\right) \approx \mathbf{51.4285714}252^\circ \approx \left(\frac{\mathbf{360}}{\mathbf{7}}\right)^\circ$$

(3) Draw the angle bisector ON_2 of the **reflex angle $\boldsymbol{\angle MON_1}$**.

$$\boldsymbol{reflex\ \angle MON_1} = 360^\circ - (\theta_1^\circ + \theta_2^\circ)$$

$$\Rightarrow \angle N_1ON_2 = \frac{reflex\ \angle MON_1}{2} = 180^\circ - \frac{(\theta_1^\circ + \theta_2^\circ)}{2}$$

$$\angle M_2ON_2 = \angle N_1ON_2 - \angle M_2ON_1 = \frac{reflex\ \angle MON_1}{2} - \angle M_2ON_1$$

$$= \left\{180^\circ - \frac{(\theta_1^\circ + \theta_2^\circ)}{2}\right\} - \{180^\circ - (\theta_1^\circ + \theta_2^\circ)\} = \frac{(\theta_1^\circ + \theta_2^\circ)}{2}$$

$$\Rightarrow \boldsymbol{\angle M_2ON_2 = \theta^\circ} = \frac{\cos^{-1}\left(\frac{275}{\sqrt{194539}}\right) + \cos^{-1}\left(\frac{1293}{2\sqrt{1075173}}\right)}{2} = \mathbf{51.4285714}2906426\ ...^\circ$$

This angle measurement is the one discussed in **Section 3** just before the explanation of the 4 constructions. It gives an extremely low error of $\mathbf{+(4.92832 \times 10^{-10})^\circ}$ or equivalently $\mathbf{+(9.58285 \times 10^{-10})\,\%}$.

> The **reflex $\boldsymbol{\angle MON_1}$** has been chosen to draw the angle bisector, because the angle bisector of **obtuse $\boldsymbol{\angle MON_1}$** will **almost** coincide with the line segment OM_1, because the difference between the angles θ_1° and θ° is so small that it will be probably visible only under a microscope. It may be noted that the points N_2, O and M_1 are **not collinear**.

(4) At this point, as a good approximation of the central angle has been obtained, so the heptagon can be constructed according to the steps discussed in **Section 2.2.1**.

Estimating the impact of this error. The error in the construction of the central angle of this heptagon is $+(4.92832 \times 10^{-10})^\circ$, implying that the constructed angle is slightly *more* than the exact expected central angle of $\left(\frac{360}{7}\right)^\circ$. So, the total error in the *full circle* is

$$\epsilon_c = 7 \times \left(4.92832 \times 10^{-10}\right)^\circ \approx \left(3.44982 \times 10^{-9}\right)^\circ \approx \left(6.02107 \times 10^{-11}\right)\ radian$$

This means that the total angle *overshoots* the complete angle of 360°. This will be visible to a normal human eye only when this overshoot is at least $\mathbf{0.1\,mm}$. The radius r of such a circle is estimated as

$$r \times \epsilon_c = 0.1\,mm$$

$$r = \frac{0.1}{\epsilon_c}\,mm = \frac{0.1}{6.02107 \times 10^{-11}}\,mm \approx \left(1.66083 \times 10^9\right)\,mm = \mathbf{1660.83\,km}$$

This means that the error in construction of the central angle is not visible to a normal human eye as long as the circumcircle of this regular heptagon has a radius smaller than 1660 km, which is *a huge number as compared to any real life drawings.*

Key findings

- Using the elementary law of cosines along with computer programming has helped to find 4 highly accurate constructions of a regular heptagon using only a marked ruler and a compass.
- This work showcases the power of applying *computer programming* to get good approximations of a regular heptagon.
- The most significant contribution of this work is the ability to **practically draw** these 4 types of regular heptagons on a A2 sheet of paper, thus making it *usable for architects and engineers in real life.*

Future work

Further research on properties of triangles coupled with computer programming may lead to discovery of more such higher precision constructions of non-constructible regular polygons. It can also lead to more results related to some other aspects of Euclidean geometry.

Disclosure statement

The authors declare that there is no conflict of interest in this research.

References

[1] Sir Hamilton WR. On Rober's Construction of The Heptagon. Philosophical Magazine, 27, 1864, p. 124-132
[2] Williamson RS. The Mathematical Gazette, Volume 39, Issue 328, May 1955, p. 134 DOI: https://doi.org/10.2307/3609979
[3] Johnson C. The Mathematical Gazette, Vol. 59, No. 407, Mar, 1975, p. 17-21
[4] Wunsche A. Applied Mathematics, 2014, 5, p. 2370-2380 DOI: http://dx.doi.org/10.4236/am.2014.515229
[5] Pfeifer D. Fundamental Journal of Mathematics and Mathematical Sciences p-ISSN: 2395-7573; e-ISSN: 2395-7581 Volume 18, Issue 2, 2024, p. 87-93