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Generalization of a Circle in the Context of a Parabola and Triangle

Benjamin Lee Warren[a] [2]
e-mail: warrenbenjamin52@yahoo.com
web: https://orcid.org/0000-0002-2829-6953

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1. Introduction

We justify the claim made in Euclid #8554.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let P be a point in the plane not on line BC.
Let w be the parabola whose focus is P and whose directrix is line BC.
Let w_b and w_c be the points on w whose tangent lines are parallel to lines AB and AC respectively.
Let the lines tangent to w at w_b and w_c intersect lines AB and AC at p_c and p_b respectively and each other at Q.
Then, given that the four points w_b, w_c, p_b, and p_c are distinct, then they are concyclic if and only if A, P, and Q are collinear.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates (x_A, y_A).
B has coordinates $(0, 0)$.
C has coordinates $(1, 0)$.
P has coordinates (x_P, y_P).
w is given by the equation $y = ((x - x_P)^2 + y_P^2)/(2 * y_P)$.
w has derivative $\frac{dy}{dx} = (x - x_P)/y_P$.
Line AB is given by the equation $y = x * y_A/x_A$.
Line AC is given by the equation $y = (x - 1) * y_A/(x_A - 1)$.

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[2]Corresponding author

w_b has coordinates
$(x_P + y_A * y_P/x_A, ((y_A^2 + x_A^2) * y_P)/(2 * x_A^2))$.
w_c has coordinates
$(x_P + y_A * y_P/(x_A - 1), (y_A^2 * y_P)/(2 * (x_A - 1)^2) + y_P/2)$.
The line tangent to w at w_b is given by the equation
$y = ((y_A^2 + x_A^2) * y_P)/(2 * x_A^2) + (x - (x_P + y_A * y_P/x_A)) * (y_A/x_A)$.
The line tangent to w at w_c is given by the equation
$y = (y_A^2 * y_P)/(2 * (x_A - 1)^2) + y_P/2 + (x - (x_P + y_A * y_P/(x_A - 1))) * (y_A/(x_A - 1))$.
p_b has coordinates
$((x_A^2 * y_P - x_A * y_P)/(2 * y_A) + (y_A * y_P)/(2 * x_A) - (y_A * y_P)/2 - x_A * x_P + x_P + x_A, -(y_A^2 * y_P)/(2 * x_A) + (x_A * y_P)/2 - x_P * y_A + y_A)$.
p_c has coordinates
$((x_A * (y_A - x_A + 1) * (y_A + x_A - 1) * y_P)/(2 * (x_A - 1) * y_A) + x_A * x_P, ((y_A - x_A + 1) * (y_A + x_A - 1) * y_P)/(2 * (x_A - 1)) + x_P * y_A)$.
Q has coordinates
$(((2 * x_A - 1) * y_A * y_P)/(2 * (x_A - 1) * x_A) + x_P, (y_A^2 * y_P)/(2 * (x_A - 1) * x_A) + y_P/2)$.
Line AQ is given by the equation
$y = y_A + (x - x_A) * ((y_A^2 * y_P)/(2 * (x_A - 1) * x_A) + y_P/2 - y_A)/(((2 * x_A - 1) * y_A * y_P)/(2 * (x_A - 1) * x_A) + x_P - x_A)$.
Points w_b, w_c, p_b, and p_c are concyclic if and only if

$$det \begin{bmatrix} x_{w_b}^2 + y_{w_b}^2 & x_{w_b} & y_{w_b} & 1 \\ x_{w_c}^2 + y_{w_c}^2 & x_{w_c} & y_{w_c} & 1 \\ x_{p_b}^2 + y_{p_b}^2 & x_{p_b} & y_{p_b} & 1 \\ x_{p_c}^2 + y_{p_c}^2 & x_{p_c} & y_{p_c} & 1 \end{bmatrix} = 0$$

$\iff$

$(((x_P+y_A*y_P/x_A)-(x_P+y_A*y_P/(x_A-1)))*(-(y_A^2*y_P)/(2*x_A)+(x_A*y_P)/2-x_P*y_A+y_A)+(((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)-(x_P+y_A*y_P/x_A))*((y_A^2*y_P)/(2*(x_A-1)^2)+y_P/2)+((x_P+y_A*y_P/(x_A-1))-((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A))*(((y_A^2+x_A^2)*y_P)/(2*x_A^2)))*(((y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1))+x_P*y_A)^2+(((x_P+y_A*y_P/(x_A-1))-(x_P+y_A*y_P/x_A))*(-(y_A^2*y_P)/(2*x_A)+(x_A*y_P)/2-x_P*y_A+y_A)^2+((x_P+y_A*y_P/x_A)-((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A))*((y_A^2*y_P)/(2*(x_A-1)^2)+y_P/2)^2+(((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)-(x_P+y_A*y_P/(x_A-1)))*(((y_A^2+x_A^2)*y_P)/(2*x_A^2))^2+((x_P+y_A*y_P/(x_A-1))-(x_P+y_A*y_P/x_A))*((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)^2+((x_P+y_A*y_P/x_A)^2-(x_P+y_A*y_P/(x_A-1))^2)*((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)+(x_P+y_A*y_P/x_A)*(x_P+y_A*y_P/(x_A-1))^2-(x_P+y_A*y_P/x_A)^2*(x_P+y_A*y_P/(x_A-1)))*(((y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1))+x_P*y_A)+(((x_P+y_A*y_P/x_A)-((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P))*((y_A^2*y_P)/(2*(x_A-1)^2)+y_P/2)+(((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)-(x_P+y_A*y_P/(x_A-1)))*(((y_A^2+x_A^2)*y_P)/(2*x_A^2)))*(-(y_A^2*y_P)/(2*x_A)+(x_A*y_P)/2-x_P*y_A+y_A)^2+((((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)-(x_P+y_A*y_P/x_A))*((y_A^2*y_P)/(2*(x_A-1)^2)+y_P/2)^2+((x_P+y_A*y_P/(x_A-1))-((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P))*(((y_A^2+x_A^2)*y_P)/(2*x_A^2))^2+((x_P+y_A*y_P/x_A)-(x_P+y_A*y_P/(x_A-1)))*((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)^2+((x_P+y_A*y_P/(x_A-1))^2-(x_P+y_A*y_P/x_A)^2)*((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)-(x_P+y_A*y_P/x_A)*(x_P+y_A*y_P/(x_A-$

$1))^2+(x_P+y_A*y_P/x_A)^2*(x_P+y_A*y_P/(x_A-1)))*(-(y_A^2*y_P)/(2*x_A)+(x_A*y_P)/2-x_P*y_A+y_A)+(((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)-((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P))*(((y_A^2+x_A^2)*y_P)/(2*x_A^2))*((y_A^2*y_P)/(2*(x_A-1)^2)+y_P/2)^2+((((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)-((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A))*(((y_A^2+x_A^2)*y_P)/(2*x_A^2))^2+(((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)-(x_P+y_A*y_P/x_A))*((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)^2+((x_P+y_A*y_P/x_A)^2-((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)^2)*((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)+(x_P+y_A*y_P/x_A)*((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)^2-(x_P+y_A*y_P/x_A)^2*((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A))*((y_A^2*y_P)/(2*(x_A-1)^2)+y_P/2)+(((x_P+y_A*y_P/(x_A-1))-((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A))*((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)^2+(((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)^2-(x_P+y_A*y_P/(x_A-1))^2)*((x_A*(y_A-x_A+1)*(y_A+x_A-1)*y_P)/(2*(x_A-1)*y_A)+x_A*x_P)-(x_P+y_A*y_P/(x_A-1))*((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A)^2+(x_P+y_A*y_P/(x_A-1))^2*((x_A^2*y_P-x_A*y_P)/(2*y_A)+(y_A*y_P)/(2*x_A)-(y_A*y_P)/2-x_A*x_P+x_P+x_A))*(((y_A^2+x_A^2)*y_P)/(2*x_A^2))=0$

$\iff$

$(y_P*((2*x_A-1)*y_A*y_P-(x_P+x_A-1)*y_A^2+(x_A-1)*x_A*x_P-x_A^3+x_A^2)*(((x_A-2)*y_A^2-x_A^3+2*x_A^2-x_A)*y_P+2*(x_A-1)^2*x_P*y_A)*(((x_A+1)*y_A^2-x_A^3+x_A^2)*y_P+2*x_A^2*(x_P-1)*y_A))/(8*(x_A-1)^4*x_A^4)=0$

$\iff$

$0=y_P*[(2*x_A-1)*y_A*y_P-(x_P+x_A-1)*y_A^2+(x_A-1)*x_A*x_P-x_A^3+x_A^2]*[((x_A-2)*y_A^2-x_A^3+2*x_A^2-x_A)*y_P+2*(x_A-1)^2*x_P*y_A]*[((x_A+1)*y_A^2-x_A^3+x_A^2)*y_P+2*x_A^2*(x_P-1)*y_A]$

$\iff$

A, P, and Q are collinear if and only if

$0=y_A+(x_P-x_A)*((y_A^2*y_P)/(2*(x_A-1)*x_A)+y_P/2-y_A)/(((2*x_A-1)*y_A*y_P)/(2*(x_A-1)*x_A)+x_P-x_A)-y_P$

$\iff$

$0=y_P*(x_A^2*x_P-y_A*y_P-x_A*x_P-x_A*y_A^2-x_P*y_A^2+x_A^2-x_A^3+y_A^2+2*x_A*y_A*y_P)/(2*x_A*x_P+y_A*y_P-2*x_A^2*x_P-2*x_A^2+2*x_A^3-2*x_A*y_A*y_P)$

$\iff$

$0=x_A^2*x_P-y_A*y_P-x_A*x_P-x_A*y_A^2-x_P*y_A^2+x_A^2-x_A^3+y_A^2+2*x_A*y_A*y_P.$

We see that if A, P, and Q are collinear then w_b, w_c, p_b, and p_c are concyclic since the two expressions $(2*x_A-1)*y_A*y_P-(x_P+x_A-1)*y_A^2+(x_A-1)*x_A*x_P-x_A^3+x_A^2$ and $x_A^2*x_P-y_A*y_P-x_A*x_P-x_A*y_A^2-x_P*y_A^2+x_A^2-x_A^3+y_A^2+2*x_A*y_A*y_P$ are equal.

However, w_b, w_c, p_b, and p_c will still be concyclic if either $0=((x_A-2)*y_A^2-x_A^3+2*x_A^2-x_A)*y_P+2*(x_A-1)^2*x_P*y_A$ or $0=((x_A+1)*y_A^2-x_A^3+x_A^2)*y_P+2*x_A^2*(x_P-1)*y_A$.

In the first case, rearranging gives $y_P=(2*x_P*y_A*(x_A-1)^2)/(x_A-y_A^2*(x_A-2)-2*x_A^2+x_A^3)$. Plugging this into w_c and p_c gives the coordinates $(x_P+y_A*((2*x_P*y_A*(x_A-1)^2)/(x_A-y_A^2*(x_A-2)-2*x_A^2+x_A^3))/(x_A-1), (y_A^2*((2*x_P*y_A*(x_A-1)^2)/(x_A-y_A^2*(x_A-2)-2*x_A^2+x_A^3)))/(2*(x_A-1)^2)+((2*x_P*y_A*(x_A-1)^2)/(x_A-y_A^2*(x_A-2)-2*x_A^2+x_A^3))/2)$ and

$((x_A*(y_A-x_A+1)*(y_A+x_A-1)*((2*x_P*y_A*(x_A-1)^2)/(x_A-y_A^2*(x_A-2)-2*x_A^2+x_A^3)))/(2*(x_A-1)*y_A)+x_A*x_P,((y_A-x_A+1)*(y_A+x_A-1)*((2*x_P*y_A*(x_A-1)^2)/(x_A-y_A^2*(x_A-2)-2*x_A^2+x_A^3)))/(2*(x_A-1))+x_P*y_A)$
$\iff$
$(-(x_A*x_P*(y_A^2+(x_A-1)^2))/((x_A-2)*y_A^2-(x_A-1)^2*x_A),-(x_P*y_A*(y_A^2+(x_A-1)^2))/((x_A-2)*y_A^2-(x_A-1)^2*x_A))$ and $(-(x_A*x_P*(y_A^2+(x_A-1)^2))/((x_A-2)*y_A^2-(x_A-1)^2*x_A),-(x_P*y_A*(y_A^2+(x_A-1)^2))/((x_A-2)*y_A^2-(x_A-1)^2*x_A))$ which are the same point.

In the second case, rearranging gives $y_P=-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)$. Plugging this into w_b and p_b gives the coordinates $(x_P+y_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3))/x_A,((y_A^2+x_A^2)*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*x_A^2))$ and $((x_A^2*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3))-x_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*y_A)+(y_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*x_A)-(y_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/2-x_A*x_P+x_P+x_A,-(y_A^2*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*x_A)+(x_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/2-x_P*y_A+y_A)$
We then see that
$x_P+y_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3))/x_A-((x_A^2*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3))-x_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*y_A)+(y_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*x_A)-(y_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/2-x_A*x_P+x_P+x_A)=0$
and $((y_A^2+x_A^2)*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*x_A^2)-(-(y_A^2*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/(2*x_A)+(x_A*(-(2*x_A^2*y_A*(x_P-1))/(y_A^2*(x_A+1)+x_A^2-x_A^3)))/2-x_P*y_A+y_A)=0$ which indicates that they are the same point, and so proves the result. □

REFERENCES

[1] Warren, B.L., "A Circle in the Context of a Triangle and Parabola", *International Journal of Computer Discovered Mathematics*, under review.

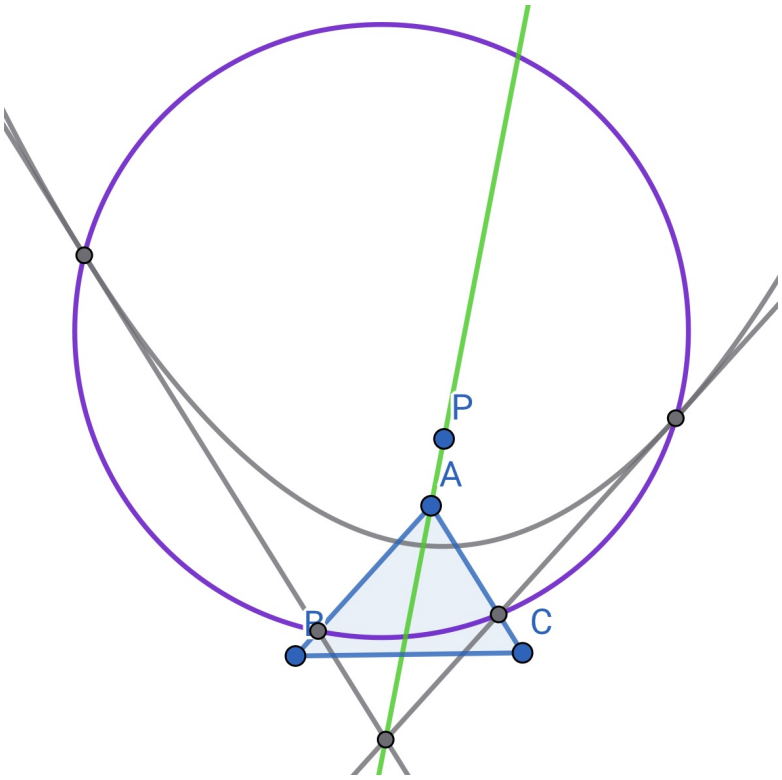

Figure 1. Illustration.