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An Equilateral Triangle With a Special Circumdiameter

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1. Introduction

We provide justification for the observation made in Euclid #8451.

2. Results

Theorem 2.1. *Let ABC be a triangle with centroid G.*
Let $A'B'C'$ be the first Napoleon triangle.
Let A'' be the reflection of A about line GA'.
Define B'' and C'' cyclically.
Let G_A be the centroid of $AA'A''$.
Define G_B and G_C cyclically.
Then,
1. the pairs $A'A'', B'B''$, and $C'C''$ have a common midpoint, P, and
2. the centroids of $AB''C'', BA''C''$, and $CA''B''$ form an equilateral triangle whose circumcircle is identical to the circle whose diameter connects the centroid of $G_AG_BG_C$ and the reflection of G about P.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
G has coordinates $((x_C + 1)/3, y_C/3)$.
A' has coordinates
$((x_C + 1 + x_C/2 + (3^{1/2} * y_C)/2 + 1/2)/3, (y_C + (y_C - 3^{1/2} * (x_C - 1))/2)/3)$.

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B' has coordinates
$((x_C + x_C/2 - (3^{1/2} * y_C)/2)/3, (y_C + y_C/2 + (3^{1/2} * x_C)/2)/3)$.
C' has coordinates
$(1/2, -3^{1/2}/6)$.
Line GA' is given by the equation
$y = (y_C/3)+(x-((x_C+1)/3))*((y_C/3)-((y_C+(y_C-3^{1/2}*(x_C-1))/2)/3))/(((x_C+1)/3) - ((x_C + 1 + x_C/2 + (3^{1/2} * y_C)/2 + 1/2)/3))$.
Line GB' is given by the equation
$y = (y_C/3)+(x-((x_C+1)/3))*((y_C/3)-((y_C+y_C/2+(3^{1/2}*x_C)/2)/3))/(((x_C+1)/3) - ((x_C + x_C/2 - (3^{1/2} * y_C)/2)/3))$.
Line GC' is given by the equation
$y = (y_C/3) + (x - ((x_C + 1)/3)) * ((y_C/3) - (-3^{1/2}/6))/(((x_C + 1)/3) - (1/2))$.
A'' has coordinates
$(2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))), 2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))$.
B'' has coordinates
$(2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1, 2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))$.
C'' has coordinates
$(2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C, 2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)$.
G_A has coordinates
$(((x_C+1+x_C/2+(3^{1/2}*y_C)/2+1/2)/3+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/3, ((y_C+(y_C-3^{1/2}*(x_C-1))/2)/3+2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/3)$.
G_B has coordinates
$((1+(x_C+x_C/2-(3^{1/2}*y_C)/2)/3+2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1)/3, ((y_C+y_C/2+(3^{1/2}*x_C)/2)/3+2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/3)$.
G_C has coordinates
$((x_C + 1/2 + 2 * ((24 * x_C^3 * y_C + 12 * 3^{1/2} * x_C^3 - 24 * x_C^2 * y_C - 12 * 3^{1/2} * x_C^2 + 24 * x_C * y_C^3 + 28 * 3^{1/2} * x_C * y_C^2 + 36 * x_C * y_C + 6 * 3^{1/2} * x_C + 4 * 3^{1/2} * y_C^2 + 12 * y_C + 3 * 3^{1/2})/(12 * (2 * y_C + 3^{1/2}) * (x_C^2 - x_C + y_C^2 + 3^{1/2} * y_C + 1))) - x_C)/3, (y_C + (-3^{1/2}/6) + 2 * ((12 * x_C^2 * y_C + 4 * 3^{1/2} * x_C^2 - 12 * x_C * y_C - 4 * 3^{1/2} * x_C + 12 * y_C^3 + 12 * 3^{1/2} * y_C^2 + 12 * y_C + 3^{1/2})/(12 * (x_C^2 - x_C + y_C^2 + 3^{1/2} * y_C + 1))) - y_C)/3)$.
The centroid of $G_AG_BG_C$ has coordinates
$((3^{1/2} * ((9 * x_C + 9) * (8 * x_C^2 - 11 * x_C + 8) + y_C * (3 * 3^{1/2} * (16 * x_C^3 - 6 * x_C^2 + 6 * x_C + 37) + y_C * (144 * x_C + 6 * 3^{1/2} * y_C * (8 * x_C + 5) + 171))))/(162 * (2 * y_C + 3^{1/2}) * (x_C^2 - x_C + y_C * (y_C + 3^{1/2}) + 1)), (3^{1/2} * (x_C * (27 * x_C - 27) + y_C * (3 * 3^{1/2} * (8 * x_C^2 - 8 * x_C + 5) + y_C * (24 * 3^{1/2} * y_C + 63))))/(162 * (x_C^2 - x_C + y_C * (y_C + 3^{1/2}) + 1)))$.
The centroid of $AB''C''$ has coordinates
$((2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*$

$y_C+1)))-x_C)/3, (2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3).$
The centroid of $BA''C''$ has coordinates
$((1+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3, (2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3).$
The centroid of $CA''B''$ has coordinates
$((x_C+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1)/3, (y_C+2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/3).$
Consider the point P, the midpoint of C' and C'' whose coordinates are given by
$((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2, (2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2).$
The reflection of G about P has coordinates
$(2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3, 2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3).$
We verify that P is the midpoint of $A'A''$ and $B'B''$ since we calculate
$((x_C+1+x_C/2+(3^{1/2}*y_C)/2+1/2)/3+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/2-((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2) = 0,$
$((y_C+(y_C-3^{1/2}*(x_C-1))/2)/3+2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/2-((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2) = 0,$
$((x_C+x_C/2-(3^{1/2}*y_C)/2)/3+2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1)/2-((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2) = 0,$
and
$((y_C+y_C/2+(3^{1/2}*x_C)/2)/3+2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/2-((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2) = 0.$

We verify that the centroids of $AB''C''$, $BA''C''$, and $CA''B''$ form an equilateral triangle by showing that their sides are the same length via the calculations

$$(((2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3)-((1+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3))^2+(((2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3)-((2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3))^2-((((2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3)-((x_C+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1)/3))^2+(((2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3)-((y_C+2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/3))^2) = 0,$$

and

$$(((2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3)-((1+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3))^2+(((2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3)-((2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3))^2-((((1+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3)-((x_C+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1)/3))^2+$$

$$(((2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3)-((y_C+2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/3))^2)=0.$$

Consider the circle whose diameter connects the centroid of $G_AG_BG_C$ and the reflection of G about P, whose equation is given by

$$(x-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2+(y-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2=0.$$

We verify that the centroids of $AB''C''$, $BA''C''$, and $CA''B''$ lie on this circle since we calculate

$$((2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2+((2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*$$

$$x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2=0,$$

$$((1+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C)/3-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2+((2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C)/3-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2=0,$$

and

$$((x_C+2*(-(3^{1/2}*(x_C^2+y_C^2-1)*(y_C-3^{1/2}*x_C+3^{1/2}))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*((3*x_C^3+3^{1/2}*x_C^2*y_C+6*x_C^2+3*x_C*y_C^2-2*3^{1/2}*x_C*y_C-12*x_C+3^{1/2}*y_C^3+12*y_C^2+12*3^{1/2}*y_C+12)/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-1)/3-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2+((y_C+2*((3^{1/2}*(x_C+3^{1/2}*y_C+1)*(x_C^2+y_C^2-1))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))+2*(((3^{1/2}*y_C-x_C+2)*(3^{1/2}*x_C^2-2*3^{1/2}*x_C+3^{1/2}*y_C^2))/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1))))/3-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*$$

$3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2)^2-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))-((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2)^2=0$
as desired. □

$(((3^{1/2}*((9*x_C+9)*(8*x_C^2-11*x_C+8)+y_C*(3*3^{1/2}*(16*x_C^3-6*x_C^2+6*x_C+37)+y_C*(144*x_C+6*3^{1/2}*y_C*(8*x_C+5)+171))))/(162*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((24*x_C^3*y_C+12*3^{1/2}*x_C^3-24*x_C^2*y_C-12*3^{1/2}*x_C^2+24*x_C*y_C^3+28*3^{1/2}*x_C*y_C^2+36*x_C*y_C+6*3^{1/2}*x_C+4*3^{1/2}*y_C^2+12*y_C+3*3^{1/2})/(12*(2*y_C+3^{1/2})*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-x_C+1/2)/2)-(x_C+1)/3)/2,((3^{1/2}*(x_C*(27*x_C-27)+y_C*(3*3^{1/2}*(8*x_C^2-8*x_C+5)+y_C*(24*3^{1/2}*y_C+63))))/(162*(x_C^2-x_C+y_C*(y_C+3^{1/2})+1))+2*((2*((12*x_C^2*y_C+4*3^{1/2}*x_C^2-12*x_C*y_C-4*3^{1/2}*x_C+12*y_C^3+12*3^{1/2}*y_C^2+12*y_C+3^{1/2})/(12*(x_C^2-x_C+y_C^2+3^{1/2}*y_C+1)))-y_C-3^{1/2}/6)/2)-y_C/3)/2).$

References

[1] Weisstein, Eric W. "Napoleon's Theorem." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/NapoleonsTheorem.html

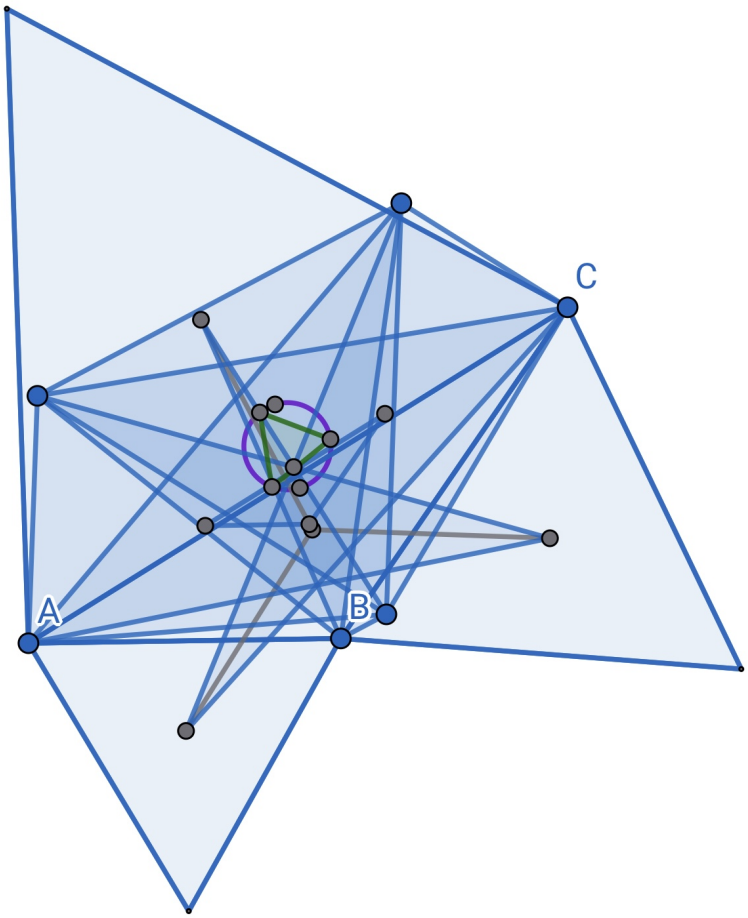

FIGURE 1. Illustration.