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A Circle with an Euler Line Diameter

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1. Introduction

We provide justification for the claim made in Euclid #8312.

2. Results

Theorem 2.1. *Let ABC be a triangle.*
Let $A'B'C'$ be the medial triangle.
Let m be the midpoint of B' and C'.
Let A_b and A_c be points on line $B'C'$ such that $|mA_b| : |mB'| = |mA_c| : |mC'| = t$.
Define B_a, B_c, C_a, and C_b cyclically.
Let $P_c(A)$ and $P_c(B)$ be the orthogonal projections of A_b and B_a respectively on line AB.
Define $P_b(A), P_b(C), P_a(B)$, and $P_a(C)$ cyclically.
Then the circumcircle of the centroids of $AP_a(B)P_a(C), BP_b(A)P_b(C)$, and $CP_c(A)P_c(B)$ is identical to the circle whose diameter is the chord connecting $X(2)$ and the point P on the Euler Line such that $|PX(5)| : |X(5)X(2)| = t$.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
A' has coordinates $((x_C + 1)/2, y_C/2)$.
B' has coordinates $(x_C/2, y_C/2)$.
C' has coordinates $(1/2, 0)$.
m has coordinates $((x_C + 1)/4, y_C/4)$.

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A_b has coordinates $(x_D, (y_C * (x_D - 1/2))/(x_C - 1))$.
A_c has coordinates $((x_C + 1)/2 - x_D, ((x_C/2 - x_D) * y_C)/(x_C - 1))$.
B_a has coordinates $((2 * x_C * x_D - 1)/(2 * x_C - 2), (y_C * (x_D - 1/2))/(x_C - 1))$.
B_c has coordinates
$(-(x_C * (2 * x_D - x_C - 1) + 1)/(2 * (x_C - 1)), ((x_C/2 - x_D) * y_C)/(x_C - 1))$.
C_a has coordinates $((2 * x_D + x_C^2 - x_C - 1)/(2 * (x_C - 1)), y_C/2)$.
C_b has coordinates $(-(-x_C^2 + 2 * x_D)/(2 * x_C - 2), y_C/2)$.
$P_a(B)$ has coordinates
$(-((2 * x_D - x_C - 2) * y_C^2 + x_C * (2 * (x_C - 1) * x_D - x_C^2 + 2) - 1)/(2 * (y_C^2 + (x_C - 1)^2)), -(y_C * ((2 * x_D - x_C) * y_C^2 + 2 * (x_C - 1) * x_C * x_D - x_C^3 + 2 * x_C^2 - 2 * x_C + 1))/(2 * (x_C - 1) * (y_C^2 + (x_C - 1)^2)))$.
$P_a(C)$ has coordinates
$(((x_C + 1) * y_C^2 - 2 * (x_C - 1) * x_D + (x_C - 1) * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2)), (y_C * (y_C^2 - 2 * x_D + x_C^2 - 2 * x_C + 2))/(2 * (y_C^2 + (x_C - 1)^2)))$.
$P_b(A)$ has coordinates
$(-(x_C * ((2 * x_D - x_C) * y_C^2 + 2 * (x_C - 1) * x_C * x_D - x_C^3 + x_C))/(2 * (x_C - 1) * (y_C^2 + x_C^2)), -(y_C * ((2 * x_D - x_C) * y_C^2 + 2 * (x_C - 1) * x_C * x_D - x_C^3 + x_C))/(2 * (x_C - 1) * (y_C^2 + x_C^2)))$.
$P_b(C)$ has coordinates
$((x_C * ((x_C - 1) * y_C^2 + x_C * (2 * x_D + x_C^2 - x_C - 1)))/(2 * (x_C - 1) * (y_C^2 + x_C^2)), (y_C * ((x_C - 1) * y_C^2 + x_C * (2 * x_D + x_C^2 - x_C - 1)))/(2 * (x_C - 1) * (y_C^2 + x_C^2)))$.
$P_c(A)$ has coordinates $(x_D, 0)$.
$P_c(B)$ has coordinates $((2 * x_C * x_D - 1)/(2 * x_C - 2), 0)$,
The centroid of ABC has coordinates $((x_C + 1)/3, y_C/3)$.
The nine point center of ABC has coordinates
$(x_C/2 + 1/4, (-x_C^2 + x_C + y_C^2)/(4 * y_C))$.
The Euler line of ABC is given by the equation
$y = (y_C/3) + (x - ((x_C + 1)/3)) * ((y_C/3) - (((1 - x_C) * x_C)/y_C))/(((x_C + 1)/3) - x_C)$.
The centroid of $AP_a(B)P_a(C)$ has coordinates
$(-((2 * x_D - 2 * x_C - 3) * y_C^2 + 2 * (x_C - 1) * (x_C + 1) * x_D - (x_C - 1) * (x_C + 1) * (2 * x_C - 1))/(6 * (y_C^2 + (x_C - 1)^2)), -(y_C * ((2 * x_D - 2 * x_C + 1) * y_C^2 + 2 * (x_C - 1) * (x_C + 1) * x_D - (x_C - 1) * (2 * x_C^2 - 3 * x_C + 3)))/(6 * (x_C - 1) * (y_C^2 + (x_C - 1)^2)))$.
The centroid of $BP_b(A)p_b(C)$ has coordinates
$(-((x_C * (2 * x_D - 2 * x_C - 1) + 2) * y_C^2 + x_C^2 * (2 * (x_C - 2) * x_D - 2 * x_C^2 - x_C + 4))/(6 * (x_C - 1) * (y_C^2 + x_C^2)), -(y_C * ((2 * x_D - 2 * x_C + 1) * y_C^2 + x_C * (2 * (x_C - 2) * x_D - 2 * x_C^2 + x_C + 2)))/(6 * (x_C - 1) * (y_C^2 + x_C^2)))$.
The centroid of $CP_c(A)P_c(B)$ has coordinates
$((x_C + x_D + (2 * x_C * x_D - 1)/(2 * x_C - 2))/3, y_C/3)$.
Consider the circle whose equation is given by
$(x - ((2 * (2 * x_C - 1) * (x_D + x_C) - 3)/(12 * (x_C - 1))))^2 + (y - (-((2 * x_D - 4 * x_C + 3) * y_C)/(12 * (x_C - 1)) - (x_C * (2 * x_D - 1))/(4 * y_C)))^2 - ((x_C + 1)/3 - ((2 * (2 * x_C - 1) * (x_D + x_C) - 3)/(12 * (x_C - 1))))^2 - (y_C/3 - (-((2 * x_D - 4 * x_C + 3) * y_C)/(12 * (x_C - 1)) - (x_C * (2 * x_D - 1))/(4 * y_C)))^2 = 0$.
We verify that the center of this circle lies on this Euler line of ABC since we calculate
$(y_C/3) + ((2 * (2 * x_C - 1) * (x_D + x_C) - 3)/(12 * (x_C - 1)) - ((x_C + 1)/3)) * ((y_C/3) - (((1 - x_C) * x_C)/y_C))/(((x_C + 1)/3) - x_C) - (-((2 * x_D - 4 * x_C + 3) * y_C)/(12 * (x_C - 1)) - (x_C * (2 * x_D - 1))/(4 * y_C)) = 0$.
We verify that the centroids of ABC, $AP_a(B)P_a(C)$, $BP_b(A)P_b(C)$, and $CP_c(A)P_c(B)$

lie on this circle since we calculate
$((x_C+1)/3-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2+(y_C/3-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2-((x_C+1)/3-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2-(y_C/3-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2=0,$
$(-((2*x_D-2*x_C-3)*y_C^2+2*(x_C-1)*(x_C+1)*x_D-(x_C-1)*(x_C+1)*(2*x_C-1))/(6*(y_C^2+(x_C-1)^2))-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2+(-(y_C*((2*x_D-2*x_C+1)*y_C^2+2*(x_C-1)*(x_C+1)*x_D-(x_C-1)*(2*x_C^2-3*x_C+3)))/(6*(x_C-1)*(y_C^2+(x_C-1)^2))-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2-((x_C+1)/3-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2-(y_C/3-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2=0,$
$((x_C*(2*x_D-2*x_C-1)+2)*y_C^2+x_C^2*(2*(x_C-2)*x_D-2*x_C^2-x_C+4))/(6*(x_C-1)*(y_C^2+x_C^2))-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2+(-(y_C*((2*x_D-2*x_C+1)*y_C^2+x_C*(2*(x_C-2)*x_D-2*x_C^2+x_C+2)))/(6*(x_C-1)*(y_C^2+x_C^2))-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2-((x_C+1)/3-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2-(y_C/3-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2=0,$
and
$((x_C+x_D+(2*x_C*x_D-1)/(2*x_C-2))/3-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2+(y_C/3-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2-((x_C+1)/3-((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1))))^2-(y_C/3-(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C)))^2=0.$
Now consider the point P on this circle which is opposite the centroid of ABC, whose coordinates are given by
$(2*((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1)))-(x_C+1)/3, 2*(-((2*x_D-4*x_C+3)*y_C)/(12*(x_C-1))-(x_C*(2*x_D-1))/(4*y_C))-y_C/3).$
We verify that $|mA_b|:|mB'|=|PX(5)|:|X(5)X(2)|$ since we calculate
$(x_m-x_{A_b})/(x_m-x_{B'})-(x_P-x_{X(5)})/(x_{X(5)}-x_{X(2)})$
$=((x_C+1)/4-x_D)/((x_C+1)/4-x_C/2)-(2*((2*(2*x_C-1)*(x_D+x_C)-3)/(12*(x_C-1)))-(x_C+1)/3-(x_C/2+1/4))/(x_C/2+1/4-(x_C+1)/3)=0$
as desired. □

References

[1] Weisstein, Eric W. "Euler Line." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/EulerLine.html

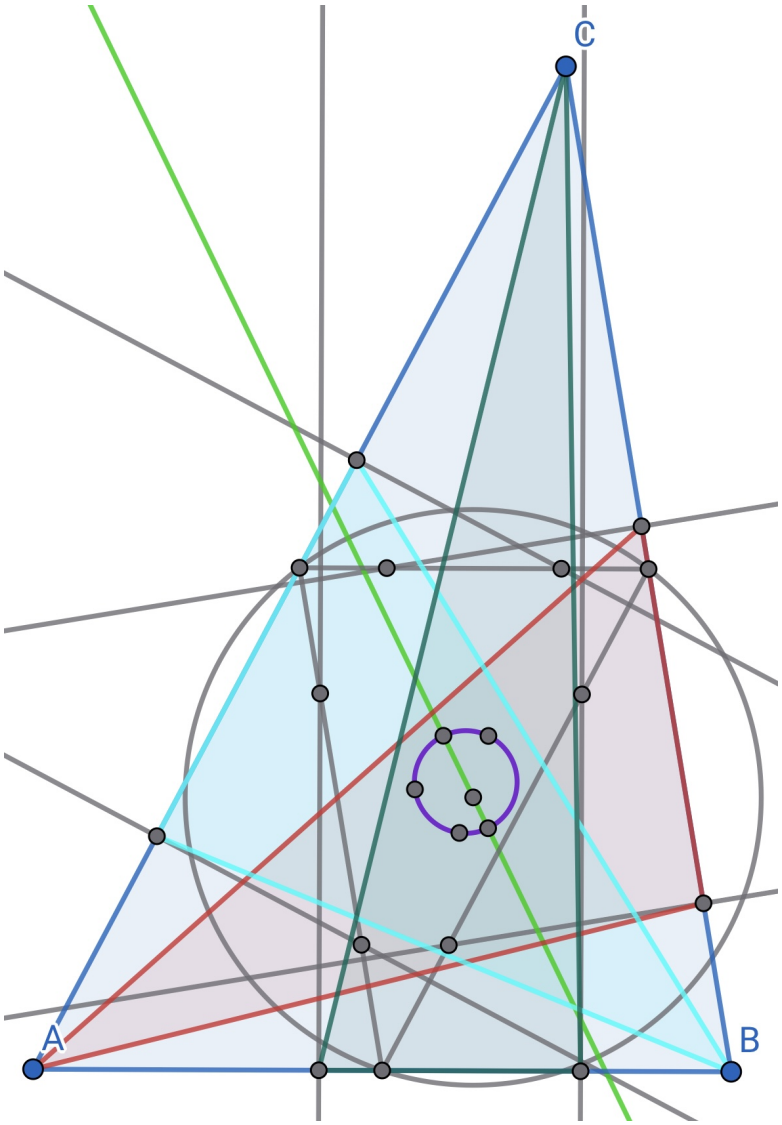

Figure 1. Illustration.