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The Circle Whose Diameter Connects the Orthocenters of a Triangle and its Orthic Triangle

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1. Introduction

We verify the claim made in Euclid #8380.

2. Results

Theorem 2.1. *Let ABC be a triangle with $A'B'C'$ the orthic triangle.*
Let m_A be the midpoint of A and the orthocenter of ABC.
Define m_B and m_C cyclically.
Let A_b be the reflection of m_A about line $A'C'$.
Define A_c, B_a, B_c, C_a, C_b cyclically.
Then lines A_bA_c, B_aB_c, and C_aC_b are concurrent at the orthocenter of $A'B'C'$ and the circumcircle of the midpoints of A_bA_c, B_aB_c, and C_aC_b is identical to the circle whose diameter connects the orthocenters of ABC and $A'B'C'$.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
A' has coordinates
$(y_C^2/(y_C^2 + (x_C - 1)^2), -((x_C - 1) * y_C)/(y_C^2 + (x_C - 1)^2))$.
B' has coordinates
$(x_C^2/(y_C^2 + x_C^2), (x_C * y_C)/(y_C^2 + x_C^2))$.
C' has coordinates $(x_C, 0)$.
Line $A'B'$ is given by the equation

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$y = -(y_C * ((2 * x_C - 1) * x - x_C))/(y_C^2 - (x_C - 1) * x_C)$.
Line $A'C'$ is given by the equation
$y = (y_C * (x - x_C))/(y_C^2 + (x_C - 1) * x_C)$.
Line $B'C'$ is given by the equation
$y = -(y_C * (x - x_C))/(y_C^2 + (x_C - 1) * x_C)$.
The orthocenter of ABC has coordinates $(x_C, x_C * (1 - x_C)/y_C)$.
The orthocenter of $A'B'C'$ has coordinates
$((y_C^2 * (y_C^2 + 2 * x_C^2 + 2 * x_C - 1) + (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), -(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
m_A has coordinates $(x_C/2, x_C * (1 - x_C)/(2 * y_C))$.
m_B has coordinates $((x_C + 1)/2, x_C * (1 - x_C)/(2 * y_C))$.
m_C has coordinates $(x_C, (y_C + x_C * (1 - x_C)/y_C)/2)$.
A_b has coordinates
$((x_C * (y_C^2 * (y_C^2 + 2 * x_C^2 - 4 * x_C + 5) + (x_C - 2) * (x_C - 1)^2 * x_C))/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), (x_C * (y_C^2 * ((x_C - 3) * y_C^2 + (x_C - 1) * (2 * x_C^2 - 4 * x_C - 1)) + (x_C - 1)^3 * x_C^2))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
A_c has coordinates
$((x_C * (y_C^2 * (y_C^2 - 2 * x_C^2 + 8 * x_C - 3) - (x_C - 1)^2 * x_C * (3 * x_C - 2)))/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), -(x_C * (y_C^2 * ((3 * x_C - 5) * y_C^2 + (x_C - 1) * (2 * x_C^2 + 1)) - (x_C - 1)^3 * x_C^2))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
B_a has coordinates
$(((x_C + 1) * y_C^2 * (y_C^2 + 2 * x_C^2 - 1) + (x_C - 1)^2 * x_C^2 * (x_C + 3))/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), ((x_C - 1) * ((x_C + 2) * y_C^4 + x_C * (2 * x_C^2 - 3) * y_C^2 + x_C^5 - 2 * x_C^4 + x_C^3))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
B_c has coordinates
$(((x_C + 1) * y_C^2 * (y_C^2 - 2 * x_C^2 + 4 * x_C - 1) - 3 * (x_C - 1)^3 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), -((x_C - 1) * ((3 * x_C + 2) * y_C^4 + x_C * (2 * x_C^2 - 4 * x_C + 3) * y_C^2 - x_C^5 + 2 * x_C^4 - x_C^3))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
C_a has coordinates
$(((x_C - 1) * y_C^4 + x_C * (2 * x_C^2 - 2 * x_C + 1) * y_C^2 + x_C^5 - x_C^4 - x_C^3 + x_C^2)/((y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), -(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
C_b has coordinates
$(((x_C + 1) * y_C^4 + x_C * (2 * x_C^2 - 2 * x_C + 1) * y_C^2 + x_C^5 - 3 * x_C^4 + 3 * x_C^3 - x_C^2)/((y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), -(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
Line A_bA_c is given by the equation
$y = (2 * y_C^2 * (y_C^2 + x_C^2) * x - x_C * y_C^4 - x_C * (2 * x_C + 1) * y_C^2 + x_C^5 - 2 * x_C^4 + x_C^3)/(2 * (x_C - 1) * y_C * (y_C^2 + x_C^2))$.
The midpoint of A_b and A_c has coordinates
$((x_C * (y_C^2 * (y_C^2 + 2 * x_C + 1) - (x_C - 1)^2 * x_C^2))/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), -((x_C - 1) * x_C * (y_C^2 * (y_C^2 + 2 * x_C + 1) - (x_C - 1)^2 * x_C^2))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
Line B_aB_c is given by the equation
$y = (y_C^2 * (2 * (y_C^2 + (x_C - 1)^2) * x - (x_C + 1) * y_C^2 - x_C + 1) + (x_C - 1)^3 * x_C^2)/(2 * x_C * y_C * (y_C^2 + (x_C - 1)^2))$.
The midpoint of B_a and B_c has coordinates
$(((x_C + 1) * y_C^2 * (y_C^2 + 2 * x_C - 1) - (x_C - 3) * (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)), -((x_C - 1) * x_C * (y_C^2 * (y_C^2 - 2 * x_C + 3) - (x_C - 1)^2 * x_C^2))/(2 *$

$y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))$.
Line C_aC_b is given by the equation
$y = -(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))$.
The midpoint of C_a and C_b has coordinates
$(x_C, (-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))+(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))))/2)$.
We verify that the orthocenter of $A'B'C'$ lies on lines A_bA_c, B_aB_c, and C_aC_b since we calculate
$(2 * y_C^2 * (y_C^2 + x_C^2) * ((y_C^2 * (y_C^2 + 2 * x_C^2 + 2 * x_C - 1) + (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))) - x_C * y_C^4 - x_C * (2 * x_C + 1) * y_C^2 + x_C^5 - 2 * x_C^4 + x_C^3)/(2 * (x_C - 1) * y_C * (y_C^2 + x_C^2)) - (-(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))) = 0$,
$(y_C^2 * (2 * (y_C^2 + (x_C - 1)^2) * ((y_C^2 * (y_C^2 + 2 * x_C^2 + 2 * x_C - 1) + (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))) - (x_C + 1) * y_C^2 - x_C + 1) + (x_C - 1)^3 * x_C^2)/(2 * x_C * y_C * (y_C^2 + (x_C - 1)^2)) - (-(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))) = 0$,
and
$-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2)) - (-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))) = 0$.
Next, consider the circle whose diameter connects the orthocenters of ABC and $A'B'C'$ whose equation is given by
$(x - (x_C + (y_C^2 * (y_C^2 + 2 * x_C^2 + 2 * x_C - 1) + (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))/2)^2 + (y - (x_C * (1 - x_C)/y_C + (-(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))))/2)^2 - (x_C - (x_C + (y_C^2 * (y_C^2 + 2 * x_C^2 + 2 * x_C - 1) + (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))/2)^2 - (x_C * (1 - x_C)/y_C - (x_C * (1 - x_C)/y_C + (-(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))))/2)^2 = 0$.
We verify that the midpoints of A_bA_c, B_aB_c, and C_aC_b lie on this circle since we calculate
$((x_C*(y_C^2*(y_C^2+2*x_C+1)-(x_C-1)^2*x_C^2))/(2*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))-(x_C+(y_C^2*(y_C^2+2*x_C^2+2*x_C-1)+(x_C-1)^2*x_C^2)/(2*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2)))/2)^2+(-((x_C-1)*x_C*(y_C^2*(y_C^2+2*x_C+1)-(x_C-1)^2*x_C^2))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))-(x_C*(1-x_C)/y_C+(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))))/2)^2-(x_C-(x_C+(y_C^2*(y_C^2+2*x_C^2+2*x_C-1)+(x_C-1)^2*x_C^2)/(2*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2)))/2)^2-(x_C*(1-x_C)/y_C-(x_C*(1-x_C)/y_C+(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))))/2)^2 = 0$,
$(((x_C + 1) * y_C^2 * (y_C^2 + 2 * x_C - 1) - (x_C - 3) * (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)) - (x_C + (y_C^2 * (y_C^2 + 2 * x_C^2 + 2 * x_C - 1) + (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))/2)^2 + (-((x_C - 1) * x_C * (y_C^2 * (y_C^2 - 2 * x_C + 3) - (x_C - 1)^2 * x_C^2))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)) - (x_C * (1 - x_C)/y_C + (-(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C - 1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))))/2)^2 - (x_C - (x_C + (y_C^2 * (y_C^2 + 2 * x_C^2 + 2 * x_C - 1) + (x_C - 1)^2 * x_C^2)/(2 * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2)))/2)^2 - (x_C * (1 - x_C)/y_C - (x_C * (1 - x_C)/y_C + (-(((y_C - 1) * y_C + (x_C - 1) * x_C) * (y_C * (y_C + 1) + (x_C -$

$1) * x_C) * (y_C^2 - (x_C - 1) * x_C))/(2 * y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_C^2))))/2)^2 = 0,$
and
$(x_C-(x_C+(y_C^2*(y_C^2+2*x_C^2+2*x_C-1)+(x_C-1)^2*x_C^2)/(2*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2)))/2)^2+((-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))+(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))))/2-(x_C*(1-x_C)/y_C+(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))))/2)^2-(x_C-(x_C+(y_C^2*(y_C^2+2*x_C^2+2*x_C-1)+(x_C-1)^2*x_C^2)/(2*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2)))/2)^2-(x_C*(1-x_C)/y_C-(x_C*(1-x_C)/y_C+(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*(y_C^2-(x_C-1)*x_C))/(2*y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_C^2))))/2)^2 = 0$
as desired. □

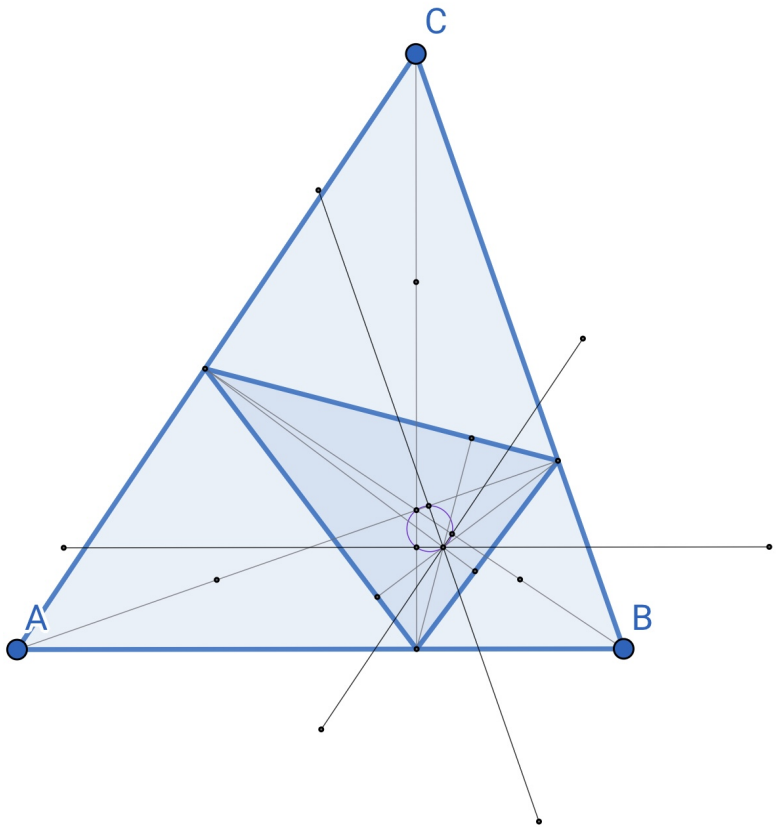

FIGURE 1. Illustration.

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