International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 10, 2025, pp. 8-10
Received December 19, 2024. Published on-line April 11, 2025
web: http://www.journal-1.eu/
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The Centroid of a Triangle Lies on a Twin Circle

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Mathematics Subject Classification (2010). 51N20.

1. Introduction

In [1], we showed that the incenter of a given triangle ABC along with the centroids of three certain triangles in reference to ABC are concyclic and that this circle is 1/3 the length of the Fuhrmann circle. Here, we will show that the centroid of a given triangle ABC along with the centroids of three other certain triangles in reference to ABC are concyclic, and that these two circles are actually twin circles.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be the point on line AC, with $\angle CAB = \angle DAB$, such that $AD = AB$.
Let E be the point on line AB, with $\angle CAB = \angle CAE$, such that $AE = AC$.
Let F be the point on line BC, with $\angle CBA = \angle FBA$, such that $BF = AB$.
Let G be the point on line AB, with $\angle CBA = \angle CBG$, such that $BG = BC$.
Let H be the point on line BC, with $\angle ACB = \angle ACH$, such that $CH = AC$.
Let I be the point on line AC, with $\angle ACB = \angle ICB$, such that $CI = BC$.
Then the centroids of ABC, AFH, BDI, and CEG are concyclic about X(5886) and the diameter of this circle is 1/3 the diameter of the Fuhrmann circle.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates $(\frac{x_C}{\sqrt{x_C^2+y_C^2}}, \frac{y_C}{\sqrt{x_C^2+y_C^2}})$.
E has coordinates $(\sqrt{x_C^2+y_C^2}, 0)$.
F has coordinates $(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1, \frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})$.
G has coordinates $(1-\sqrt{(x_C-1)^2+y_C^2}, 0)$.
H has coordinates $(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}, y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))$.
I has coordinates $(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}), y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))$.
The centroid of ABC has coordinates $((x_C+1)/3, y_C/3)$.
The centroid of AFH has coordinates
$((\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1+x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/3, (\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/3)$.
The centroid of BDI has coordinates
$((1+\frac{x_C}{\sqrt{x_C^2+y_C^2}}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/3, (\frac{y_C}{\sqrt{x_C^2+y_C^2}}+y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/3)$.
The centroid of CEG has coordinates
$((x_C+\sqrt{x_C^2+y_C^2}+1-\sqrt{(x_C-1)^2+y_C^2})/3, y_C/3)$.
The nine point center of ABC has coordinates
$(\frac{1}{2}*x_C+\frac{1}{4}, \frac{y_C^2-x_C^2+x_C}{4y_C})$.
Consider the circle whose equation is given by
$(x-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2-(((x_C+1)/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2)=0$.
We verify that the centroids of ABC, AFH, BDI, and CEG lie on this circle since we calculate that
$((x_C+1)/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2-(((x_C+1)/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2)=0$,
$((\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1+x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2-(((x_C+1)/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2)=0$,
$((1+\frac{x_C}{\sqrt{x_C^2+y_C^2}}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+((\frac{y_C}{\sqrt{x_C^2+y_C^2}}+y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+$

$2*y_C^2)/(6*y_C)))^2-(((x_C+1)/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2)=0$,
and
$((x_C+\sqrt{x_C^2+y_C^2}+1-\sqrt{(x_C-1)^2+y_C^2})/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2-(((x_C+1)/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2)=0$
as desired.

$(((((x_C+1)/3-((2/3)*(x_C+1)+((y_C^2+x_C^2)^{1/2}-(y_C^2+(x_C-1)^2)^{1/2}-2*x_C-2)/6))^2+(y_C/3-((2/3)*y_C-((y_C^2+(x_C-1)^2)^{1/2}*((y_C^2+x_C^2)^{1/2}-x_C)+(x_C-1)*(y_C^2+x_C^2)^{1/2}+2*y_C^2)/(6*y_C)))^2))/(((x_C-(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}))^2+(y_C-(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}))^2)))^{1/2}=1/3$ as desired. □

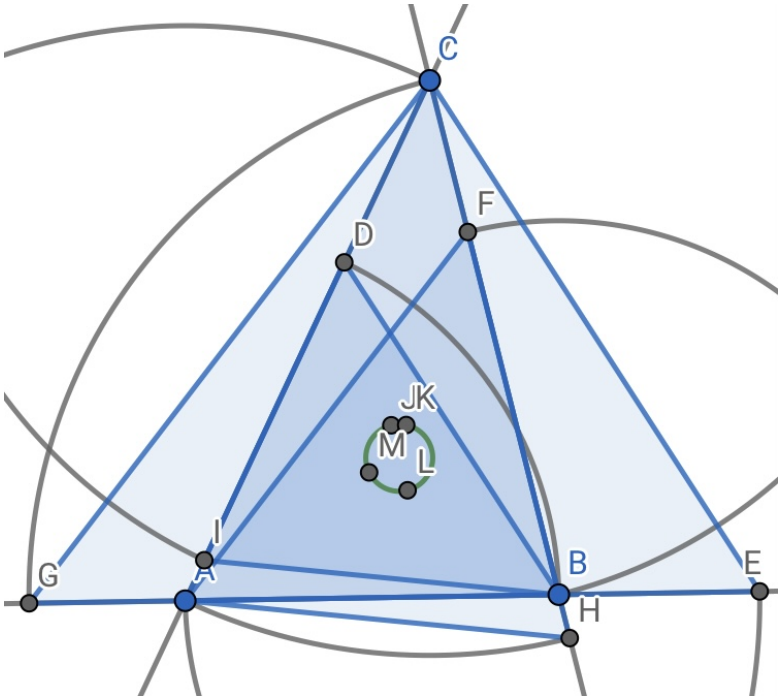

Figure 1. Illustration.

References

[1] Warren, B.L., "The Incenter Lies on a Circle." *International Journal of Computer Discovered Mathematics*, Under Review