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The Incenter as a Homothetic Center of the Intouch Triangle

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1. Results

Theorem 1.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be the point on line AC, with $\angle CAB = \angle DAB$, such that $AD = AB$.
Let E be the point on line AB, with $\angle CAB = \angle CAE$, such that $AE = AC$.
Let F be the point on line BC, with $\angle CBA = \angle FBA$, such that $BF = AB$.
Let G be the point on line AB, with $\angle CBA = \angle CBG$, such that $BG = BC$.
Let H be the point on line BC, with $\angle ACB = \angle ACH$, such that $CH = AC$.
Let I be the point on line AC, with $\angle ACB = \angle ICB$, such that $CI = BC$.
Let M be the circumcenter of A, G, and I.
Let N be the circumcenter of C, D, and F.
Let O be the circumcenter of B, E, and H.
Then triangle MNO and the intouch triangle of ABC are homothetic about the incenter of ABC.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates $(\frac{x_C}{\sqrt{x_C^2+y_C^2}}, \frac{y_C}{\sqrt{x_C^2+y_C^2}})$.
E has coordinates $(\sqrt{x_C^2 + y_C^2}, 0)$.
F has coordinates $(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}} + 1, \frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})$.

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G has coordinates $(1-\sqrt{(x_C-1)^2+y_C^2},0)$.
H has coordinates $(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}, y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))$.
I has coordinates $(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}), y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))$.
Let J be the vertex of the intouch triangle of ABC on side AB.
J has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2}+\sqrt{x_C^2+y_C^2})/2, 0)$.
Let K be the vertex of the intouch triangle of ABC on side AC.
K has coordinates $(\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}, \frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$.
Let L be the vertex of the intouch triangle of ABC on side BC.
L has coordinates $((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2, (\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)$.
M has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
N has coordinates $(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}, -\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})$.
O has coordinates $((1+\sqrt{x_C^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
Let P be the incenter of ABC.
P has coordinates $(\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}, \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})$.
We verify that lines JK and NO are parallel since we calculate
$(y_J-y_K)/(x_J-x_K)-(y_N-y_O)/(x_N-x_O)$
$=((0)-(\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}))/(((1-\sqrt{(x_C-1)^2+y_C^2}+\sqrt{x_C^2+y_C^2})/2)-$
$(\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}))-((-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})$
$-(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))/((\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2})-((1+$
$\sqrt{x_C^2+y_C^2})/2))=0$.
We verify that lines JL and MN are parallel since we calculate
$(y_J-y_L)/(x_J-x_L)-(y_M-y_N)/(x_M-x_N)$
$=((0)-((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2))/(((1-\sqrt{(x_C-1)^2+y_C^2}+$
$\sqrt{x_C^2+y_C^2})/2)-((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2))-((-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$
$-(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}))/(((1-\sqrt{(x_C-1)^2+y_C^2})/2)$
$-(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}))=0$.
We verify that lines KL and MO are parallel since we calculate
$(y_K-y_L)/(x_K-x_L)-(y_M-y_O)/(x_M-x_O)$
$=((\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$
$-((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2))/((\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})-((1+$
$x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2))$
$-((-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$
$-(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))/(((1-\sqrt{(x_C-1)^2+y_C^2})/2)-((1+$

$\sqrt{x_C^2+y_C^2})/2)) = 0$.
This confirms that triangle MNO and the intouch triangle of ABC are homothetic.
Line JN is given by the equation
$x = \frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}$.
Line KO is given by the equation
$y = (\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$
$+ (x - (\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})) * ((\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$
$-(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))/((\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})-((1+$
$\sqrt{x_C^2+y_C^2})/2))$.
Line LM is given by the equation
$y = ((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)+(x-((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2))*$
$(((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)-(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))/(((1+$
$x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2)-((1-\sqrt{(x_C-1)^2+y_C^2})/2))$.
We verify that P lies on all these lines since we calculate
$\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}-(\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}) = 0$
$(\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$
$+((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})-(\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}))*((\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$
$-(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))/((\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})-((1+$
$\sqrt{x_C^2+y_C^2})/2)) - (\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})$
$= 0$,
and
$((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}} + y_C(1 - \frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2) + (\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}} - ((1 + x_C +$
$\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2)) * (((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}} + y_C(1 - \frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2) -$
$(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))/(((1 + x_C + \frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2) -$
$((1 - \sqrt{(x_C-1)^2+y_C^2})/2)) - (\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})$
$= 0$ as desired. □

REFERENCES

[1] Warren, B.L., "Three Congruent Circles, Six Parallel Lines, X(104), and a New Construction of the Intouch Triangle", *International Journal of Computer Discovered Mathematics*, Under Review.

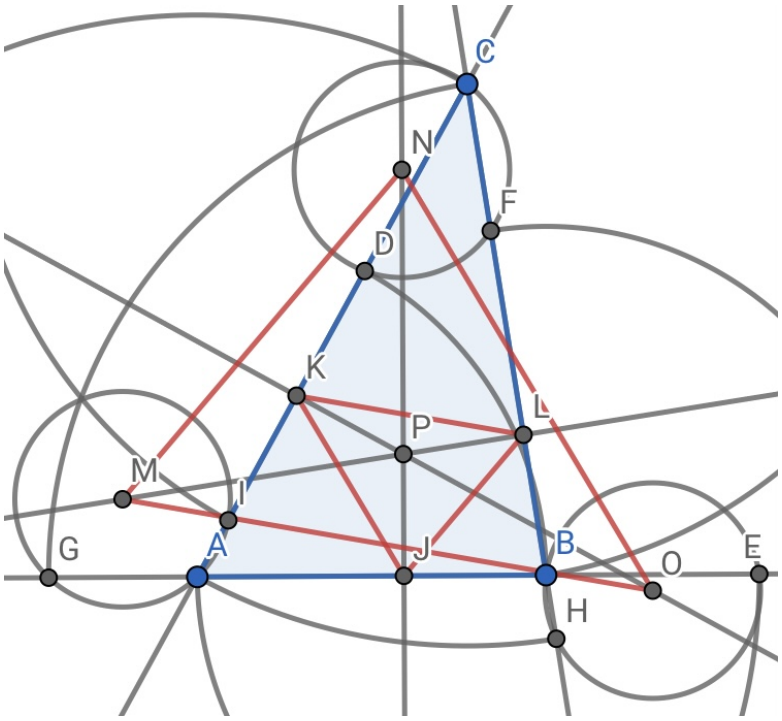

FIGURE 1. Illustration.