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Coaxal Circles in the Context of the Circumcevian Triangle

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1. Introduction

In a given triangle ABC with a point D in the plane, form the circle which passes through a triangle vertex, the circumcenter, and the corresponding vertex of the circumcevian triangle. Form the other two circles cyclically. We demonstrate here that these three circles are coaxal.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point in the plane.
Let E, F, and G be the vertices of the circumcevian triangle of D adjacent to sides AB, AC, and BC respectively.
Let H be the circumcenter of ABC.
Then circles AGH, BFH, and CEH are coaxal.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates
$(((x_C * y_D - x_D * y_C) * (y_C * (y_D - y_C) + (x_C - 1) * (x_D - x_C)))/(y_C * ((y_D - y_C)^2 + (x_D - x_C)^2)), (((x_C - 1) * y_D - (x_D - 1) * y_C) * (x_C * y_D - x_D * y_C))/(y_C * ((y_D - y_C)^2 + (x_D - x_C)^2)))$.

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F has coordinates
$((y_D*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))/(y_C*(y_D^2+(x_D-1)^2)), (y_D*((y_C^2+(x_C-1)*x_C)*y_D-(x_D-1)*y_C))/(y_C*(y_D^2+(x_D-1)^2))).$
G has coordinates
$((x_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2)), (y_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))).$
H has coordinates $(1/2, (y_C^2+(x_C-1)*x_C)/(2*y_C))$.
Circle AGH is given by the equation
$(x-(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*y_D+2*x_D*y_C^3+2*(x_C-1)*x_C*x_D*y_C)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2+(y-((2*y_C*(y_C^2+(x_C-1)*x_C)*y_D-x_D*y_C^4-(2*x_C^2-2*x_C-1)*x_D*y_C^2-(x_C-1)^2*x_C^2*x_D)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2-(((-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*y_D+2*x_D*y_C^3+2*(x_C-1)*x_C*x_D*y_C)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2+(((2*y_C*(y_C^2+(x_C-1)*x_C)*y_D-x_D*y_C^4-(2*x_C^2-2*x_C-1)*x_D*y_C^2-(x_C-1)^2*x_C^2*x_D)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2) = 0.$
Circle BFH is given by the equation
$(x-(((y_C^4+(2*x_C^2-2*x_C+3)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D+2*(x_D-1)*y_C^3+2*(x_C-1)*x_C*(x_D-1)*y_C)/(4*y_C*(y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^2+x_C))))^2+(y-((y_C*(2*y_C^2*y_D+2*x_C^2*y_D-2*x_C*y_D+x_D*y_C^3-y_C^3+2*x_C^2*x_D*y_C-2*x_C*x_D*y_C-x_D*y_C-2*x_C^2*y_C+2*x_C*y_C+y_C)+(x_C-1)^2*x_C^2*(x_D-1))/(4*y_C*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))))^2-((1-(((y_C^4+(2*x_C^2-2*x_C+3)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D+2*(x_D-1)*y_C^3+2*(x_C-1)*x_C*(x_D-1)*y_C)/(4*y_C*(y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^2+x_C))))^2+(((y_C*(2*y_C^2*y_D+2*x_C^2*y_D-2*x_C*y_D+x_D*y_C^3-y_C^3+2*x_C^2*x_D*y_C-2*x_C*x_D*y_C-x_D*y_C-2*x_C^2*y_C+2*x_C*y_C+y_C)+(x_C-1)^2*x_C^2*(x_D-1))/(4*y_C*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))))^2) = 0.$
Circle CEH is given by the equation
$(x-(((y_C^4+(2*x_C^2+2*x_C-1)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D-y_C^5-(2*(x_D+x_C^2)-1)*y_C^3+(x_C-1)*x_C*(2*x_D-x_C*(x_C+1))*y_C)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2+(y-((2*(2*x_C-1)*y_C*(y_C^2+(x_C-1)*x_C)*y_D-(3*x_D+x_C-2)*y_C^4-((2*x_C^2-2*x_C+1)*x_D+x_C*(2*x_C^2-4*x_C+1))*y_C^2+(x_C-1)^2*x_C^2*x_D-x_C^5+2*x_C^4-x_C^3)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2-((x_C-(((y_C^4+(2*x_C^2+2*x_C-1)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D-y_C^5-(2*(x_D+x_C^2)-1)*y_C^3+(x_C-1)*x_C*(2*x_D-x_C*(x_C+1))*y_C)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2+(y_C-((2*(2*x_C-1)*y_C*(y_C^2+(x_C-1)*x_C)*y_D-(3*x_D+x_C-2)*y_C^4-((2*x_C^2-2*x_C+1)*x_D+x_C*(2*x_C^2-4*x_C+1))*y_C^2+(x_C-1)^2*x_C^2*x_D-x_C^5+2*x_C^4-x_C^3)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2) = 0.$
Consider the line whose equation is given by
$y_C*(y_D*(2*x-1)-y_C*x+x_D*y_C-2*x_D*y+y)-(x_C-1)*x_C*(x-x_D) = 0.$
We verify that this line is the radical line of all three circles since we calculate
$(((2*y_C*(y_C*y_D-x_D*y_C^2-x_C^2*x_D+x_C*x_D)*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1))))/(((y_C^2+(x_C-1)^2)*(y_C^2+x_C^2)*y_D))*((x-(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*y_D+2*x_D*y_C^3+2*(x_C-1)*x_C*x_D*y_C)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2+(y-((2*y_C*(y_C^2+(x_C-1)*x_C)*y_D-x_D*y_C^4-(2*x_C^2-2*x_C-1)*x_D*y_C^2-(x_C-1)^2*x_C^2*x_D)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2-(((-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*y_D+2*x_D*y_C^3+2*(x_C-1)*x_C*x_D*y_C)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2+(((2*y_C*(y_C^2+(x_C-1)*x_C)*y_D-x_D*y_C^4-(2*x_C^2-2*x_C-1)*x_D*y_C^2-(x_C-1)^2*x_C^2*x_D)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2)-((x-(((y_C^4+$

$(2*x_C^2-2*x_C+3)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D+2*(x_D-1)*y_C^3+2*(x_C-1)*x_C*(x_D-1)*y_C)/(4*y_C*(y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^2+x_C))))^2+(y-((y_C*(2*y_C^2*y_D+2*x_C^2*y_D-2*x_C*y_D+x_D*y_C^3-y_C^3+2*x_C^2*x_D*y_C-2*x_C*x_D*y_C-x_D*y_C-2*x_C^2*y_C+2*x_C*y_C+y_C)+(x_C-1)^2*x_C^2*(x_D-1))/(4*y_C*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))))^2-((1-(((y_C^4+(2*x_C^2-2*x_C+3)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D+2*(x_D-1)*y_C^3+2*(x_C-1)*x_C*(x_D-1)*y_C)/(4*y_C*(y_C*y_D+(x_D-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^2+x_C))))^2+(((y_C*(2*y_C^2*y_D+2*x_C^2*y_D-2*x_C*y_D+x_D*y_C^3-y_C^3+2*x_C^2*x_D*y_C-2*x_C*x_D*y_C-x_D*y_C-2*x_C^2*y_C+2*x_C*y_C+y_C)+(x_C-1)^2*x_C^2*(x_D-1))/(4*y_C*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))))^2)))-(y_C*(y_D*(2*x-1)-y_C*x+x_D*y_C-2*x_D*y+y)-(x_C-1)*x_C*(x-x_D))=0$

and

$(((2*y_C*(y_C*y_D-x_D*y_C^2-x_C^2*x_D+x_C*x_D)*(y_C*((2*x_C-1)*y_D-(x_D+x_C-1)*y_C)+(x_C-1)*x_C*(x_D-x_C))))/((y_C^2+(x_C-1)^2)*(y_C^2+x_C^2)*(x_C*y_D-x_D*y_C)))*((x-(-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*y_D+2*x_D*y_C^3+2*(x_C-1)*x_C*x_D*y_C)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2+(y-((2*y_C*(y_C^2+(x_C-1)*x_C)*y_D-x_D*y_C^4-(2*x_C^2-2*x_C-1)*x_D*y_C^2-(x_C-1)^2*x_C^2*x_D)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2-(((-(((y_C-1)*y_C+(x_C-1)*x_C)*(y_C*(y_C+1)+(x_C-1)*x_C)*y_D+2*x_D*y_C^3+2*(x_C-1)*x_C*x_D*y_C)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2+(((2*y_C*(y_C^2+(x_C-1)*x_C)*y_D-x_D*y_C^4-(2*x_C^2-2*x_C-1)*x_D*y_C^2-(x_C-1)^2*x_C^2*x_D)/(4*y_C*(y_C*y_D-x_D*y_C^2-(x_C-1)*x_C*x_D))))^2)-((x-(((y_C^4+(2*x_C^2+2*x_C-1)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D-y_C^5-(2*(x_D+x_C^2)-1)*y_C^3+(x_C-1)*x_C*(2*x_D-x_C*(x_C+1))*y_C)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2+(y-((2*(2*x_C-1)*y_C*(y_C^2+(x_C-1)*x_C)*y_D-(3*x_D+x_C-2)*y_C^4-((2*x_C^2-2*x_C+1)*x_D+x_C*(2*x_C^2-4*x_C+1))*y_C^2+(x_C-1)^2*x_C^2*x_D-x_C^5+2*x_C^4-x_C^3)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2-((x_C-(((y_C^4+(2*x_C^2+2*x_C-1)*y_C^2+x_C^4-2*x_C^3+x_C^2)*y_D-y_C^5-(2*(x_D+x_C^2)-1)*y_C^3+(x_C-1)*x_C*(2*x_D-x_C*(x_C+1))*y_C)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2+(y_C-((2*(2*x_C-1)*y_C*(y_C^2+(x_C-1)*x_C)*y_D-(3*x_D+x_C-2)*y_C^4-((2*x_C^2-2*x_C+1)*x_D+x_C*(2*x_C^2-4*x_C+1))*y_C^2+(x_C-1)^2*x_C^2*x_D-x_C^5+2*x_C^4-x_C^3)/(4*y_C*((2*x_C-1)*y_C*y_D-(x_D+x_C-1)*y_C^2+(x_C-1)*x_C*x_D-x_C^3+x_C^2))))^2)))-(y_C*(y_D*(2*x-1)-y_C*x+x_D*y_C-2*x_D*y+y)-(x_C-1)*x_C*(x-x_D))=0$

as desired. □

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[2] Weisstein, Eric W. "Coaxal Circles." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/CoaxalCircles.html

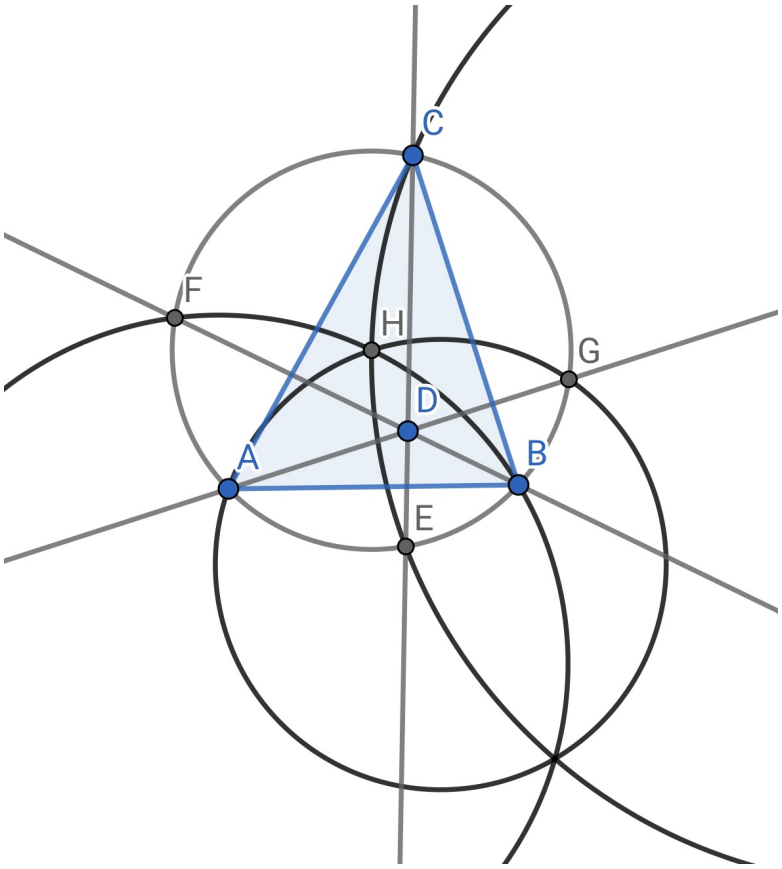

FIGURE 1. Illustration.