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Generalization of the Perspectrix Formed by a Triangle and a Cevian Triangle

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1. Introduction

Whenever two triangles are in perspective, there will exist a line which is formed by the points of intersection of the extended sides of the triangles as described in Desargues' theorem. This line is known as the perspectrix. We give here a generalization of this line in the context of when one of the two triangles is a cevian triangle.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D and E be points on line AB, F and G be points on line AC, and H and I be points on line BC such that D, E, F, G, H, and I lie on a conic section.
Let line AB intersect line FI at J.
Let line BC intersect line EG at K.
Let line AC intersect line DH at L.
Then J, K, and L are collinear.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates $(x_D, 0)$.
E has coordinates $(x_E, 0)$.
F has coordinates $(x_F, x_F * y_C/x_C)$.

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G has coordinates $(x_G, x_G * y_C/x_C)$.
H has coordinates $(x_H, y_C * (x_H - 1)/(x_C - 1))$.
The conic section passing through D, E, F, G, and H is given by the equation

$$det \begin{bmatrix} x^2 & y^2 & x*y & x & y & 1 \\ x_D^2 & y_D^2 & x_D*y_D & x_D & y_D & 1 \\ x_E^2 & y_E^2 & x_E*y_E & x_E & y_E & 1 \\ x_F^2 & y_F^2 & x_F*y_F & x_F & y_F & 1 \\ x_G^2 & y_G^2 & x_G*y_G & x_G & y_G & 1 \\ x_H^2 & y_H^2 & x_H*y_H & x_H & y_H & 1 \end{bmatrix} = 0$$

$\iff x*(x_F*x_G*(x_D+x_E)*(x_C-x_H)*(x_H-1)*y_C^2+y*(x_H*(x_G*(x_F*(x_D*(x_C-1)+x_E*(x_C-1)+2*x_C^2)+x_D*x_E*(x_C-x_C^2))-2*x_C^2*x_D*x_E+x_D*x_E*x_F*(x_C-x_C^2))-x_G*(x_F*(x_C^2+x_C*x_D*(x_C-1)+x_E*(x_C-1)*(x_C+x_D-x_C*x_D))-x_C*x_D*x_E*(x_C-1))+x_H^2*(x_D*x_E*x_C^2-2*x_F*x_G*x_C+x_F*x_G)+x_C^2*x_D*x_E+x_C*x_D*x_E*x_F*(x_C-1))*y_C)+x_C*y^2*(x_C-1)*(x_F*x_G*x_H^2-x_C*x_D*x_E*x_F-x_C*x_D*x_E*x_G+x_C*x_D*x_E*x_H+x_C*x_D*x_F*x_G+x_C*x_E*x_F*x_G+x_D*x_E*x_F*x_G-x_C*x_F*x_G*x_H-x_D*x_F*x_G*x_H-x_E*x_F*x_G*x_H-x_C*x_D*x_E*x_H^2-x_C*x_D*x_E*x_F*x_G+x_C*x_D*x_E*x_F*x_H+x_C*x_D*x_E*x_G*x_H)-x^2*x_F*x_G*y_C^2*(x_C-x_H)*(x_H-1)+x_C*y*y_C*(x_C-x_H)*(x_H-1)*(x_D*x_E*x_F+x_D*x_E*x_G-x_D*x_F*x_G-x_E*x_F*x_G)-x_D*x_E*x_F*x_G*y_C^2*(x_C-x_H)*(x_H-1)=0.$
I has coordinates
$(-((((((x_C-1)*x_D-x_C)*x_E-x_C*x_D+x_C)*x_F+x_C*x_D*x_E)*x_G+x_C*x_D*x_E*x_F-x_C^2*x_D*x_E)*x_H+((((1-x_C^2)*x_D+x_C^2)*x_E+x_C^2*x_D-x_C^2)*x_F-x_C*x_D*x_E)*x_G-x_C*x_D*x_E*x_F+x_C^2*x_D*x_E)/((((x_E+x_D-1)*x_F-x_C*x_D*x_E)*x_G-x_C*x_D*x_E*x_F+x_C^2*x_D*x_E)*x_H+((((x_C-1)*x_D-x_C)*x_E-x_C*x_D+x_C)*x_F+x_C*x_D*x_E)*x_G+x_C*x_D*x_E*x_F-x_C^2*x_D*x_E), -((x_D-1)*(x_E-1)*x_F*x_G*(x_H-x_C)*y_C)/((((x_E+x_D-1)*x_F-x_C*x_D*x_E)*x_G-x_C*x_D*x_E*x_F+x_C^2*x_D*x_E)*x_H+((((x_C-1)*x_D-x_C)*x_E-x_C*x_D+x_C)*x_F+x_C*x_D*x_E)*x_G+x_C*x_D*x_E*x_F-x_C^2*x_D*x_E))$.
Line AB is given by the equation $y = 0$.
Line AC is given by the equation $y = x * y_C/x_C$.
Line BC is given by the equation $y = y_C * (x - 1)/(x_C - 1)$.
Line FI is given by the equation
$y = (x_F*y_C*((((x_E+x_D-1)*x_G-x_C*x_D*x_E)*x_H+(((x_C-1)*x_D-x_C)*x_E-x_C*x_D+x_C)*x_G+x_C*x_D*x_E)*x-x_D*x_E*(x_G-x_C)*x_H+x_D*x_E*x_G-x_C*x_D*x_E))/(x_C*((((x_E+x_D-1)*x_F-x_D*x_E)*x_G-x_C*x_D*x_E*x_F+x_C*x_D*x_E)*x_H+((((x_C-1)*x_D-x_C)*x_E-x_C*x_D+x_C)*x_F+x_D*x_E)*x_G+x_C*x_D*x_E*x_F-x_C*x_D*x_E))$.
Line EG is given by the equation
$y = (x_G*y_C*(x-x_E))/(x_C*(x_G-x_E))$.
Line DH is given by the equation
$y = ((x_H-1)*y_C*(x-x_D))/((x_C-1)*(x_H-x_D))$.
J has coordinates
$(-(x_D*x_E*(x_C-x_G)*(x_H-1))/(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E), 0)$.
K has coordinates
$(-(x_C*x_E-x_G*(x_C-x_E*(x_C-1)))/(x_G-x_C*x_E), -((x_E-1)*x_G*y_C)/(x_G-x_C*x_E))$.
L has coordinates
$((x_C*x_D*(x_H-1))/(x_H+(x_C-1)*x_D-x_C), (x_D*(x_H-1)*y_C)/(x_H+(x_C-$

$1) * x_D - x_C))$.
Consider the line whose equation is given by
$y = (x_G*y_C*(x*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)+x_D*x_E*(x_C-x_G)*(x_H-1))*(x_E-1))/((x_C*x_E-x_G*(x_C-x_E*(x_C-1)))*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)-x_D*x_E*(x_G-x_C*x_E)*(x_C-x_G)*(x_H-1))$.
We verify that J, K, and L lie on this line since we calculate that
$(x_G*y_C*((-(x_D*x_E*(x_C-x_G)*(x_H-1))/(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E))*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)+x_D*x_E*(x_C-x_G)*(x_H-1))*(x_E-1))/((x_C*x_E-x_G*(x_C-x_E*(x_C-1)))*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)-x_D*x_E*(x_G-x_C*x_E)*(x_C-x_G)*(x_H-1))-(0) = 0,$
$(x_G*y_C*((-(x_C*x_E-x_G*(x_C-x_E*(x_C-1)))/(x_G-x_C*x_E))*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)+x_D*x_E*(x_C-x_G)*(x_H-1))*(x_E-1))/((x_C*x_E-x_G*(x_C-x_E*(x_C-1)))*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)-x_D*x_E*(x_G-x_C*x_E)*(x_C-x_G)*(x_H-1))-(-((x_E-1)*x_G*y_C)/(x_G-x_C*x_E)) = 0,$
and
$(x_G*y_C*(((x_C*x_D*(x_H-1))/(x_H+(x_C-1)*x_D-x_C))*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)+x_D*x_E*(x_C-x_G)*(x_H-1))*(x_E-1))/((x_C*x_E-x_G*(x_C-x_E*(x_C-1)))*(x_H*(x_G*(x_D+x_E-1)-x_C*x_D*x_E)-x_G*(x_C*x_D-x_C+x_E*(x_C-x_D*(x_C-1)))+x_C*x_D*x_E)-x_D*x_E*(x_G-x_C*x_E)*(x_C-x_G)*(x_H-1))-((x_D*(x_H-1)*y_C)/(x_H+(x_C-1)*x_D-x_C)) =$
0 as desired. □

The perspectrix formed by a given triangle and a cevian triangle is the special case of this where we take $D = E, F = G$, and $H = I$.

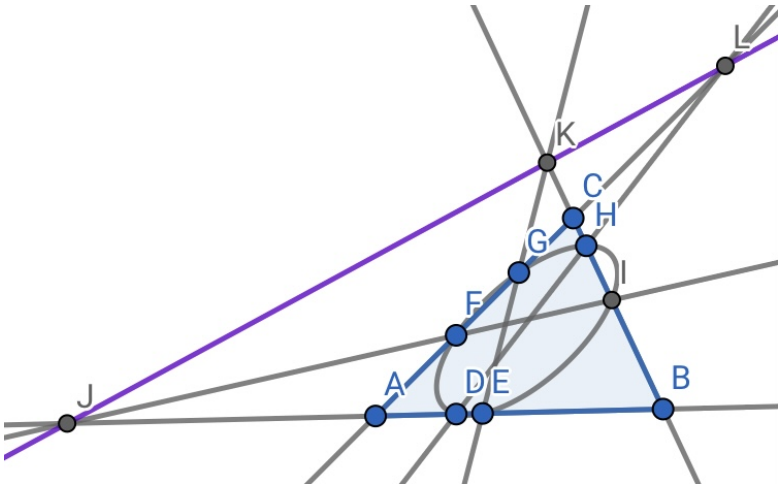

Figure 1. Illustration.

References

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