International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 10, 2025, pp. 1-3
Received December 10, 2024. Published on-line April 11, 2025
web: http://www.journal-1.eu/
©The Author(s) This article is published with open access[1].

The Midpoint of the Incenter and the Circumcenter is the Center of a Conic Section

Benjamin Lee Warren[a] [2]
University of Binghamton, Mathematics,
4400 Vestel Pkwy E, Binghamton, NY 13902
e-mail: warrenbenjamin52@yahoo.com
web: https://orcid.org/0000-0002-2829-6953

Mathematics Subject Classification (2010). 51N20.

1. Results

Theorem 1.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be the point on line AC, with $\angle CAB = \angle DAB$, such that $AD = AB$.
Let E be the point on line AB, with $\angle CAB = \angle CAE$, such that $AE = AC$.
Let F be the point on line BC, with $\angle CBA = \angle FBA$, such that $BF = AB$.
Let G be the point on line AB, with $\angle CBA = \angle CBG$, such that $BG = BC$.
Let H be the point on line BC, with $\angle ACB = \angle ACH$, such that $CH = AC$.
Let I be the point on line AC, with $\angle ACB = \angle ICB$, such that $CI = BC$.
Let M be the circumcenter of A, G, and I.
Let N be the circumcenter of C, D, and F.
Let O be the circumcenter of B, E, and H.
Let P be the incenter of ABC.
Let Q be the circumcenter of ABC.
Let R be the midpoint of P and Q.
Then J, K, L, M, N, and O lie on a conic section such that R is the midpoint of the pairs of points JN, KO, and LM.

[1]This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.
[2]Corresponding author

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates $(\frac{x_C}{\sqrt{x_C^2+y_C^2}}, \frac{y_C}{\sqrt{x_C^2+y_C^2}})$.
E has coordinates $(\sqrt{x_C^2+y_C^2}, 0)$.
F has coordinates $(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1, \frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})$.
G has coordinates $(1-\sqrt{(x_C-1)^2+y_C^2}, 0)$.
H has coordinates $(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}, y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))$.
I has coordinates $(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}), y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))$.
J has coordinates $(1/2, -\frac{(\sqrt{(x_C-1)^2+y_C^2})*(\sqrt{x_C^2+y_C^2})-x_C*(x_C-1)-y_C^2}{2y_C})$.
K has coordinates $(-\frac{\sqrt{(x_C-1)^2+y_C^2}-1}{2}, \frac{x_C*(x_C+\sqrt{(x_C-1)^2+y_C^2}-1)+y_C^2}{2y_C})$.
L has coordinates $(\frac{1+\sqrt{x_C^2+y_C^2}}{2}, -(-\frac{y_C}{2}+\frac{(x_C-1)*(\sqrt{x_C^2+y_C^2}-x_C)}{2y_C}))$.
M has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
N has coordinates $(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}, -\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})$.
O has coordinates $((1+\sqrt{x_C^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
P has coordinates $(\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}, \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})$.
Q has coordinates $(1/2, \frac{x_C(x_C-1)+y_C^2}{2y_C})$.
R has coordinates
$((1/2+\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2, (\frac{x_C(x_C-1)+y_C^2}{2y_C}+\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2)$.
We verify that J, K, L, M, N, and O lie on a conic section since we substitute the above values into the following matrix and calculate that

$$det\begin{bmatrix} x_J^2 & y_J^2 & x_J*y_J & x_J & y_J & 1 \\ x_K^2 & y_K^2 & x_K*y_K & x_K & y_K & 1 \\ x_L^2 & y_L^2 & x_L*y_L & x_L & y_L & 1 \\ x_M^2 & y_M^2 & x_M*y_M & x_M & y_M & 1 \\ x_N^2 & y_N^2 & x_N*y_N & x_N & y_N & 1 \\ x_O^2 & y_O^2 & x_O*y_O & x_O & y_O & 1 \end{bmatrix} = 0$$

and we verify that R is the midpoint of the pairs of points JN, KO, and LM since we calculate
$(1/2)(x_J+x_N)-x_R$
$=(1/2)((1/2)+(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}))-((1/2+\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2)$
$=0$.
$(1/2)(y_J+y_N)-y_R$
$=(1/2)((-\frac{(\sqrt{(x_C-1)^2+y_C^2})*(\sqrt{x_C^2+y_C^2})-x_C*(x_C-1)-y_C^2}{2y_C})+(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}))-$
$((\frac{x_C(x_C-1)+y_C^2}{2y_C}+\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2)$
$=0$.
$(1/2)(x_K+x_O)-x_R$

$= (1/2)((-\frac{\sqrt{(x_C-1)^2+y_C^2}-1}{2})+((1+\sqrt{x_C^2+y_C^2})/2))-((1/2+\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2)$
$= 0\ (1/2)(y_K + y_O) - y_R$
$= (1/2)((\frac{x_C*(x_C+\sqrt{(x_C-1)^2+y_C^2}-1)+y_C^2}{2y_C})+(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))-$
$((\frac{x_C(x_C-1)+y_C^2}{2y_C} + \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2)$
$= 0\ (1/2)(x_L + x_M) - x_R$
$= (1/2)((\frac{1+\sqrt{x_C^2+y_C^2}}{2})+((1-\sqrt{(x_C-1)^2+y_C^2})/2))-((1/2+\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2)$
$= 0,$
and
$(1/2)(y_L + y_M) - y_R$
$= (1/2)(-(-\frac{y_C}{2}+\frac{(x_C-1)*(\sqrt{x_C^2+y_C^2}-x_C)}{2y_C})+(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))-$
$((\frac{x_C(x_C-1)+y_C^2}{2y_C} + \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})/2)$
$= 0$ as desired. □

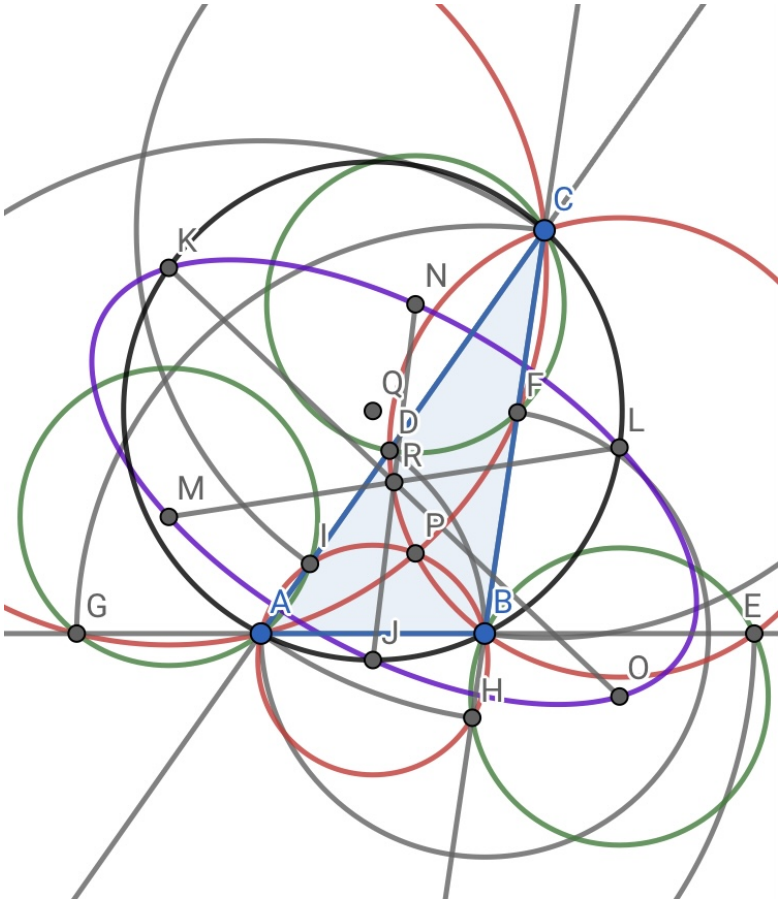

FIGURE 1. Illustration.

REFERENCES

[1] Warren, B.L., "Three Congruent Circles, Six Parallel Lines, X(104), and a New Construction of the Intouch Triangle." *International Journal of Computer Discovered Mathematics*