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The Circle Centered at the Incenter Which Passes Through the Circumcenter

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1. Introduction

We show that three certain nontrivial points lie on the circle which is centered at the incenter and which passes through the circumcenter of a triangle.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let J be the circumcenter of ABC.
Let K be the incenter of ABC.
Let α be the point on line AC with $\angle BA\alpha = \pi - \angle BAC$, such that $A\alpha = AB$.
Let β be the point on line AB with $\angle CA\beta = \pi - \angle BAC$, such that $A\beta = AC$.
Let γ be the point on line BC with $\angle \gamma CA = \pi - \angle ACB$, such that $C\gamma = AC$.
Let δ be the point on line AC with $\angle \delta CB = \pi - \angle ACB$, such that $C\delta = CB$.
Let ε be the point on line AB with $\angle \varepsilon BC = \pi - \angle ABC$, such that $\varepsilon B = BC$.
Let ζ be the point on line BC with $\angle \zeta BA = \pi - \angle ABC$, such that $\zeta B = AB$.
Let η be the circumcenter of A, α, and β.
Let θ be the circumcenter of C, γ, and δ.
Let ι be the circumcenter of B, ε, and ζ.
Let η' be the reflection of η about A.
Let θ' be the reflection of θ about C.
Let ι' be the reflection of ι about B.
Then J, η', θ', and ι' are concyclic about K.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
J has coordinates $(1/2, \frac{x_C(x_C-1)+y_C^2}{2y_C})$.
K has coordinates
$(\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}, \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})$.
α has coordinates
$(-x_C/(x_C^2+y_C^2)^{1/2}, -y_C/(x_C^2+y_C^2)^{1/2})$.
β has coordinates
$(-(x_C^2+y_C^2)^{1/2}, 0)$.
γ has coordinates
$(x_C + ((x_C^2+y_C^2)^{1/2} * (x_C-1))/((x_C-1)^2+y_C^2)^{1/2}, (x_C + ((x_C^2+y_C^2)^{1/2} * (x_C - 1))/((x_C-1)^2+y_C^2)^{1/2} - 1) * (y_C/(x_C-1)))$.
δ has coordinates
$((x_C * (((x_C^2+y_C^2) * (x_C^2 - 2*x_C + y_C^2 + 1))^{1/2} + x_C^2 + y_C^2))/(x_C^2+y_C^2), ((x_C * (((x_C^2+y_C^2) * (x_C^2 - 2*x_C + y_C^2 + 1))^{1/2} + x_C^2 + y_C^2))/(x_C^2+y_C^2)) * (y_C/x_C))$.
ε has coordinates
$(1 + ((x_C-1)^2+y_C^2)^{1/2}, 0)$.
ζ has coordinates
$(-((x_C-1) * (((x_C-1)^2+y_C^2)^{1/2} - x_C + 1) - y_C^2)/((x_C-1)^2+y_C^2), (-((x_C-1) * (((x_C-1)^2+y_C^2)^{1/2} - x_C + 1) - y_C^2)/((x_C-1)^2+y_C^2) - 1) * (y_C/(x_C-1)))$.
η has coordinates
$(-(x_C^2+y_C^2)^{1/2}/2, ((x_C-1) * (x_C^2+y_C^2)^{1/2})/(2*y_C))$.
θ has coordinates
$(x_C, y_C + ((x_C^2+y_C^2)^{1/2} * ((x_C-1)^2+y_C^2)^{1/2})/(2*y_C))$.
ι has coordinates
$((2 + ((x_C-1)^2+y_C^2)^{1/2})/2, -(x_C * ((x_C-1)^2+y_C^2)^{1/2})/(2*y_C))$.
η' has coordinates
$((x_C^2+y_C^2)^{1/2}/2, -((x_C-1) * (x_C^2+y_C^2)^{1/2})/(2*y_C))$.
θ' has coordinates
$(x_C, y_C - ((x_C^2+y_C^2)^{1/2} * ((x_C-1)^2+y_C^2)^{1/2})/(2*y_C))$.
ι' has coordinates
$((2 - ((x_C-1)^2+y_C^2)^{1/2})/2, (x_C * ((x_C-1)^2+y_C^2)^{1/2})/(2*y_C))$.
The circle centered at K which passes through J is given by the equation
$(x - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((1/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{x_C(x_C-1)+y_C^2}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0$.
We verify that η', θ', and ι' lie on this circle since we calculate that
$((x_C^2+y_C^2)^{1/2}/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (-((x_C-1) * (x_C^2+y_C^2)^{1/2})/(2*y_C) - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((1/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{x_C(x_C-1)+y_C^2}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0$,
$(x_C - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y_C - ((x_C^2+y_C^2)^{1/2} * ((x_C-1)^2+y_C^2)^{1/2})/(2*y_C) - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((1/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{x_C(x_C-1)+y_C^2}{2y_C} -$

$\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$

$((2 - ((x_C - 1)^2 + y_C^2)^{1/2})/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + ((x_C * ((x_C - 1)^2 + y_C^2)^{1/2})/(2 * y_C) - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((1/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{x_C(x_C-1)+y_C^2}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0$ as desired. □

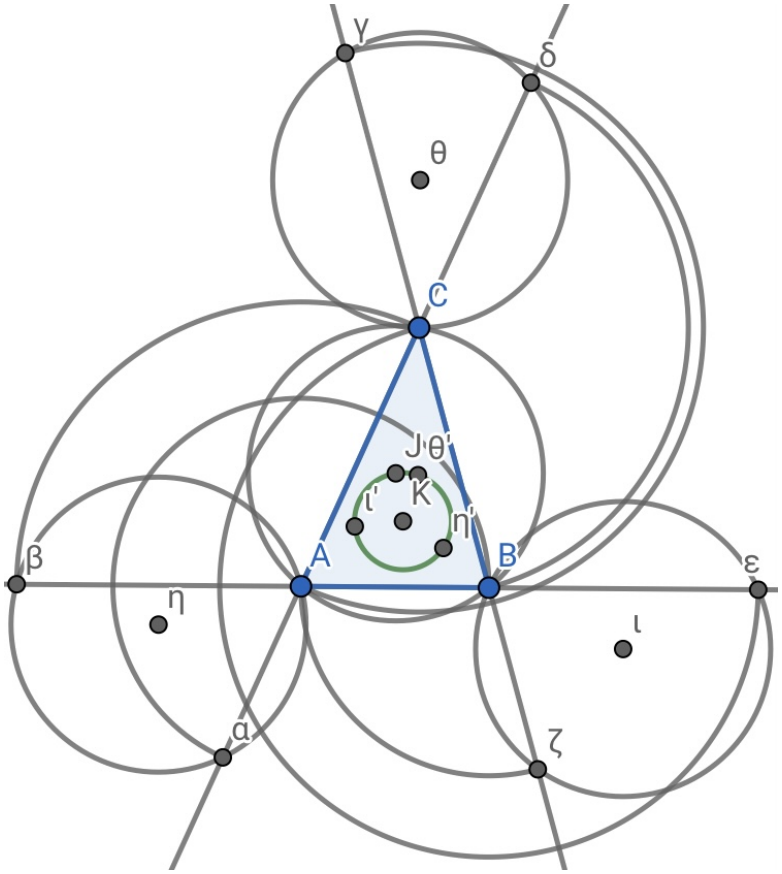

Figure 1. Illustration.

References

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