International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 9, 2024, pp. 507-511.
Received November 12, 2024. Published on-line December 31, 2024.
web: http://www.journal-1.eu/
©The Author(s) This article is published with open access[1].

A Result Related to a Generalization of the Orthocenter

Benjamin Lee Warren[a] [2]
University of Binghamton, Mathematics,
4400 Vestel Pkwy E, Binghamton, NY 13902
e-mail: warrenbenjamin52@yahoo.com
web: https://orcid.org/0000-0002-2829-6953

Mathematics Subject Classification (2010). 51N20.

1. Introduction

In [1] we gave a generalization of the orthocenter of a triangle. Here, we will prove a result in the same configuration which is related to it.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let P be the incenter of ABC.
Let J be a point in the plane different than P and not on a side of ABC.
Let α be the point on line AC with $\angle BA\alpha = \pi - \angle BAC$, such that $A\alpha = AB$.
Let β be the point on line AB with $\angle CA\beta = \pi - \angle BAC$, such that $A\beta = AC$.
Let γ be the point on line BC with $\angle \gamma CA = \pi - \angle ACB$, such that $C\gamma = AC$.
Let δ be the point on line AC with $\angle \delta CB = \pi - \angle ACB$, such that $C\delta = CB$.
Let ε be the point on line AB with $\angle \varepsilon BC = \pi - \angle ABC$, such that $\varepsilon B = BC$.
Let ζ be the point on line BC with $\angle \zeta BA = \pi - \angle ABC$, such that $\zeta B = AB$.
Let η be the circumcenter of A, α, and β.
Let θ be the circumcenter of C, γ, and δ.
Let ι be the circumcenter of B, ε, and ζ.
Let κ be a point on line AJ.
Let λ be the point on line BJ such that $B\lambda : J\lambda = A\kappa : J\kappa$.
Let μ be the point on line CJ such that $C\mu : J\mu = A\kappa : J\kappa$.

[1]This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.
[2]Corresponding author

Then if any three of the four lines $\eta\kappa, \theta\mu$, $\iota\lambda$, and JP are concurrent then all four lines are concurrent.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
P has coordinates
$(\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}, \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})$.
J has coordinates (x_J, y_J) which is different than P.
α has coordinates
$(-x_C/(x_C^2 + y_C^2)^{1/2}, -y_C/(x_C^2 + y_C^2)^{1/2})$.
β has coordinates
$(-(x_C^2 + y_C^2)^{1/2}, 0)$.
γ has coordinates
$(x_C + ((x_C^2 + y_C^2)^{1/2} * (x_C - 1))/((x_C - 1)^2 + y_C^2)^{1/2}, (x_C + ((x_C^2 + y_C^2)^{1/2} * (x_C - 1))/((x_C - 1)^2 + y_C^2)^{1/2} - 1) * (y_C/(x_C - 1)))$.
δ has coordinates
$((x_C * (((x_C^2 + y_C^2) * (x_C^2 - 2 * x_C + y_C^2 + 1))^{1/2} + x_C^2 + y_C^2))/(x_C^2 + y_C^2), ((x_C * (((x_C^2 + y_C^2) * (x_C^2 - 2 * x_C + y_C^2 + 1))^{1/2} + x_C^2 + y_C^2))/(x_C^2 + y_C^2)) * (y_C/x_C))$.
ε has coordinates
$(1 + ((x_C - 1)^2 + y_C^2)^{1/2}, 0)$.
ζ has coordinates
$(-((x_C - 1) * (((x_C - 1)^2 + y_C^2)^{1/2} - x_C + 1) - y_C^2)/((x_C - 1)^2 + y_C^2), (-((x_C - 1) * (((x_C - 1)^2 + y_C^2)^{1/2} - x_C + 1) - y_C^2)/((x_C - 1)^2 + y_C^2) - 1) * (y_C/(x_C - 1)))$.
η has coordinates
$(-(x_C^2 + y_C^2)^{1/2}/2, ((x_C - 1) * (x_C^2 + y_C^2)^{1/2})/(2 * y_C))$.
θ has coordinates
$(x_C, y_C + ((x_C^2 + y_C^2)^{1/2} * ((x_C - 1)^2 + y_C^2)^{1/2})/(2 * y_C))$.
ι has coordinates
$((2 + ((x_C - 1)^2 + y_C^2)^{1/2})/2, -(x_C * ((x_C - 1)^2 + y_C^2)^{1/2})/(2 * y_C))$.
κ has coordinates
$(x_k, x_k * y_J/x_J)$.
λ has coordinates
$(x_k - \frac{x_k}{x_J} + 1, \frac{x_k * y_J}{x_J})$.
μ has coordinates
$(x_C + (\frac{x_k}{x_J})(x_J - x_C), y_C + (\frac{x_k}{x_J})(y_J - y_C))$.
Line $\eta\kappa$ is given by the equation
$y = (x_k * y_J/x_J) + (x - x_k) * (((((x_C - 1) * (x_C^2 + y_C^2)^{1/2})/(2 * y_C)) - (x_k * y_J/x_J))/((-(x_C^2 + y_C^2)^{1/2}/2) - x_k))$.
Line $\theta\mu$ is given by the equation
$y = (y_C + ((x_C^2 + y_C^2)^{1/2} * ((x_C - 1)^2 + y_C^2)^{1/2})/(2 * y_C)) + (x - x_C) * (((y_C + (\frac{x_k}{x_J})(y_J - y_C)) - (y_C + ((x_C^2 + y_C^2)^{1/2} * ((x_C - 1)^2 + y_C^2)^{1/2})/(2 * y_C)))/((x_C + (\frac{x_k}{x_J})(x_J - x_C)) - x_C))$.
Line $\iota\lambda$ is given by the equation
$y = (-(x_C * ((x_C - 1)^2 + y_C^2)^{1/2})/(2 * y_C)) + (x - ((2 + ((x_C - 1)^2 + y_C^2)^{1/2})/2)) * (((-(x_C * ((x_C - 1)^2 + y_C^2)^{1/2})/(2 * y_C)) - (\frac{x_k * y_J}{x_J}))/(((2 + ((x_C - 1)^2 + y_C^2)^{1/2})/2) - (x_k - \frac{x_k}{x_J} + 1)))$.
Line JP is given by the equation

$y = y_J + (x - x_J) * (((\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}) - y_J)/((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}) - x_J))$.
Let $Q = (((((x_C-1)*(x_C^2+y_C^2)^{1/2})/(2*y_C)) - (x_k*y_J/x_J))/((-(x_C^2+y_C^2)^{1/2}/2) - x_k))$.
Let $R = -x_k * (((((x_C-1)*(x_C^2+y_C^2)^{1/2})/(2*y_C)) - (x_k*y_J/x_J))/((-(x_C^2+y_C^2)^{1/2}/2) - x_k)) + (x_k*y_J/x_J)$.
Let $S = (((y_C + (\frac{x_k}{x_J})(y_J - y_C)) - (y_C + ((x_C^2+y_C^2)^{1/2} * ((x_C-1)^2+y_C^2)^{1/2})/(2*y_C)))/((x_C + (\frac{x_k}{x_J})(x_J - x_C)) - x_C))$.
Let $T = -x_C*(((y_C+(\frac{x_k}{x_J})(y_J-y_C))-(y_C+((x_C^2+y_C^2)^{1/2}*((x_C-1)^2+y_C^2)^{1/2})/(2*y_C)))/((x_C+(\frac{x_k}{x_J})(x_J-x_C))-x_C))+(y_C+((x_C^2+y_C^2)^{1/2}*((x_C-1)^2+y_C^2)^{1/2})/(2*y_C))$.
Let $U = (((-(x_C * ((x_C-1)^2+y_C^2)^{1/2})/(2*y_C)) - (\frac{x_k*y_J}{x_J}))/(((2 + ((x_C-1)^2 + y_C^2)^{1/2})/2) - (x_k - \frac{x_k}{x_J} + 1)))$.
Let $V = -((2+((x_C-1)^2+y_C^2)^{1/2})/2) * (((-(x_C*((x_C-1)^2+y_C^2)^{1/2})/(2*y_C)) - (\frac{x_k*y_J}{x_J}))/(((2 + ((x_C-1)^2 + y_C^2)^{1/2})/2) - (x_k - \frac{x_k}{x_J} + 1))) + (-(x_C * ((x_C-1)^2 + y_C^2)^{1/2})/(2*y_C))$.
Let $W = (((\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}) - y_J)/((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}}) - x_J))$.
Let $Z = -x_J*(((\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})-y_J)/((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})-x_J))+y_J$.

Lines $\eta\kappa, \theta\mu$, and $\iota\lambda$ are concurrent if and only if

$$det \begin{bmatrix} Q & -1 & R \\ S & -1 & T \\ U & -1 & V \end{bmatrix} = 0$$

$\iff Q*(T-V) + S*V - T*U + R*(U-S) = 0.$
Lines $\eta\kappa, \theta\mu$, and JP are concurrent if and only if

$$det \begin{bmatrix} Q & -1 & R \\ S & -1 & T \\ W & -1 & Z \end{bmatrix} = 0$$

$\iff Q*(T-Z) + S*Z - T*W + R*(W-S) = 0.$
We have that $Q*(T-V) + S*V - T*U + R*(U-S)$
$= -((x_J-x_k)*(((x_C-1)^2+y_C^2)^{1/2}*(x_J^2*(-x_C^4+x_C^3)-x_J*(-x_C^4+x_C^3)+y_C^2*(x_k*(-2*x_C*x_J^2+2*x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^2+x_C)))-y_J*((x_C^2+y_C^2)^{1/2}*(y_C*(x_J*(x_C^3-2*x_C^2+x_C)-x_J*x_k*(2*x_C^2-4*x_C+2))+y_C^3*(x_k*(2*x_J-2)+x_C*x_J)+x_k*y_C*(2*x_C-2*x_J)*((x_C-1)^2+y_C^2)^{1/2})+((x_C-1)^2+y_C^2)^{1/2}*((2*x_J*x_k+x_J*(x_C-1))*y_C^3+(-x_k*(2*x_C^2*x_J-2*x_C^2)-x_J*(x_C^2-x_C^3))*y_C))+(x_C^2+y_C^2)^{1/2}*(((x_C-1)^2+y_C^2)^{1/2}*(y_C^2*(-x_J^2+x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^3+x_C^2))+y_C^2*(x_J^2*(-x_C^2+x_C)+x_k*((2-2*x_C)*x_J^2+(2*x_C-2)*x_J))+x_J^2*(-x_C^4+3*x_C^3-3*x_C^2+x_C))+y_J^2*(x_k*y_C^2*(x_C^2+y_C^2)^{1/2}*(2*x_C-2)+2*x_C*x_k*y_C^2*((x_C-1)^2+y_C^2)^{1/2})))/(x_J*y_C^2*(x_k*(2*x_J-2)-x_J*((x_C-1)^2+y_C^2)^{1/2})*(2*x_k+(x_C^2+y_C^2)^{1/2})*(x_C-x_J))$.
If we assumed that $Q*(T-V) + S*V - T*U + R*(U-S) = 0$ when $x_J = x_k$, we would then have the constraint
$0 = ((x_C-1)^2+y_C^2)^{1/2}*(x_J^2*(-x_C^4+x_C^3)-x_J*(-x_C^4+x_C^3)+y_C^2*(x_k*(-2*x_C*x_J^2+2*x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^2+x_C)))-y_J*((x_C^2+y_C^2)^{1/2}*(y_C*(x_J*(x_C^3-$

$2*x_C^2+x_C)-x_J*x_k*(2*x_C^2-4*x_C+2))+y_C^3*(x_k*(2*x_J-2)+x_C*x_J)+x_k*y_C*(2*x_C-2*x_J)*((x_C-1)^2+y_C^2)^{1/2})+((x_C-1)^2+y_C^2)^{1/2}*((2*x_J*x_k+x_J*(x_C-1))*y_C^3+(-x_k*(2*x_C^2*x_J-2*x_C^2)-x_J*(x_C^2-x_C^3))*y_C))+(x_C^2+y_C^2)^{1/2}*(((x_C-1)^2+y_C^2)^{1/2}*(y_C^2*(-x_J^2+x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^3+x_C^2))+y_C^2*(x_J^2*(-x_C^2+x_C)+x_k*((2-2*x_C)*x_J^2+(2*x_C-2)*x_J))+x_J^2*(-x_C^4+3*x_C^3-3*x_C^2+x_C))+y_J^2*(x_k*y_C^2*(x_C^2+y_C^2)^{1/2}*(2*x_C-2)+2*x_C*x_k*y_C^2*((x_C-1)^2+y_C^2)^{1/2})$.
We also have that $Q*(T-Z)+S*Z-T*W+R*(W-S)$
$=-((x_C^2+y_C^2)^{1/2}*(x_J-x_k)*(((x_C-1)^2+y_C^2)^{1/2}*(x_J^2*(-x_C^4+x_C^3)-x_J*(-x_C^4+x_C^3)+y_C^2*(x_k*(-2*x_C*x_J^2+2*x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^2+x_C)))-y_J*((x_C^2+y_C^2)^{1/2}*(y_C*(x_J*(x_C^3-2*x_C^2+x_C)-x_J*x_k*(2*x_C^2-4*x_C+2))+y_C^3*(x_k*(2*x_J-2)+x_C*x_J)+x_k*y_C*(2*x_C-2*x_J)*((x_C-1)^2+y_C^2)^{1/2})+((x_C-1)^2+y_C^2)^{1/2}*((2*x_J*x_k+x_J*(x_C-1))*y_C^3+(-x_k*(2*x_C^2*x_J-2*x_C^2)-x_J*(x_C^2-x_C^3))*y_C))+(x_C^2+y_C^2)^{1/2}*(((x_C-1)^2+y_C^2)^{1/2}*(y_C^2*(-x_J^2+x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^3+x_C^2))+y_C^2*(x_J^2*(-x_C^2+x_C)+x_k*((2-2*x_C)*x_J^2+(2*x_C-2)*x_J))+x_J^2*(-x_C^4+3*x_C^3-3*x_C^2+x_C))+y_J^2*(x_k*y_C^2*(x_C^2+y_C^2)^{1/2}*(2*x_C-2)+2*x_C*x_k*y_C^2*((x_C-1)^2+y_C^2)^{1/2})))/(2*x_J*x_k*y_C^2*(2*x_k+(x_C^2+y_C^2)^{1/2})*(x_C-x_J)*(x_J-x_C+x_J*((x_C-1)^2+y_C^2)^{1/2}+(x_C^2+y_C^2)^{1/2}*(x_J-1)))$.
If we assumed that $Q*(T-Z)+S*Z-T*W+R*(W-S)=0$ when $x_J=x_k$ then we would have the constraint
$0=((x_C-1)^2+y_C^2)^{1/2}*(x_J^2*(-x_C^4+x_C^3)-x_J*(-x_C^4+x_C^3)+y_C^2*(x_k*(-2*x_C*x_J^2+2*x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^2+x_C)))-y_J*((x_C^2+y_C^2)^{1/2}*(y_C*(x_J*(x_C^3-2*x_C^2+x_C)-x_J*x_k*(2*x_C^2-4*x_C+2))+y_C^3*(x_k*(2*x_J-2)+x_C*x_J)+x_k*y_C*(2*x_C-2*x_J)*((x_C-1)^2+y_C^2)^{1/2})+((x_C-1)^2+y_C^2)^{1/2}*((2*x_J*x_k+x_J*(x_C-1))*y_C^3+(-x_k*(2*x_C^2*x_J-2*x_C^2)-x_J*(x_C^2-x_C^3))*y_C))+(x_C^2+y_C^2)^{1/2}*(((x_C-1)^2+y_C^2)^{1/2}*(y_C^2*(-x_J^2+x_C*x_J)+x_J^2*(-x_C^2+x_C)-x_J*(-x_C^3+x_C^2))+y_C^2*(x_J^2*(-x_C^2+x_C)+x_k*((2-2*x_C)*x_J^2+(2*x_C-2)*x_J))+x_J^2*(-x_C^4+3*x_C^3-3*x_C^2+x_C))+y_J^2*(x_k*y_C^2*(x_C^2+y_C^2)^{1/2}*(2*x_C-2)+2*x_C*x_k*y_C^2*((x_C-1)^2+y_C^2)^{1/2})$.
Both of these constraints are the same, so the theorem follows. □

References

[1] Warren, B.L., "A New Generalization of the Orthocenter", *International Journal of Computer Discovered Mathematics*, Under Review.

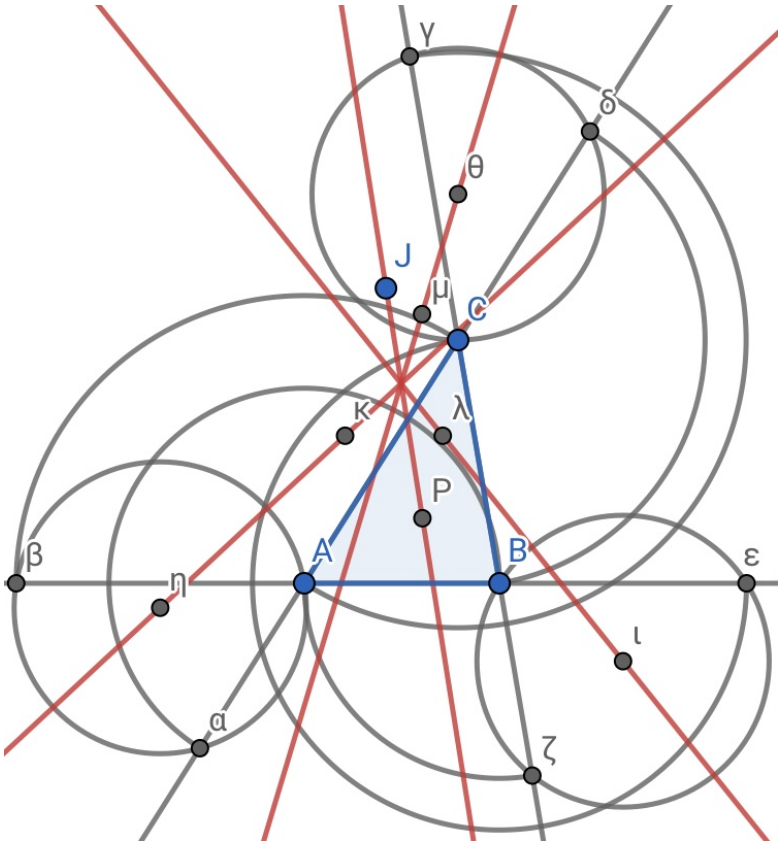

Figure 1. Illustration.