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Tangents Through Four Points on a Conic Section

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Abstract. A partial proof is given regarding a previously stated theorem about tangents of points on a given conic.

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1. Introduction

Pick a given conic section, i.e., a circle, ellipse, parabola, or hyperbola. Pick four distinct points on that conic, and form the tangent lines to those points along with the lines connecting the four points. If one examines all the points on the plane where lines intersect, it will be found that some of those points are nontrivially collinear. This was discussed in [1] but without details. Here, we give a proof of this result for the specific cases where the conic section is a parabola or a hyperbola.

2. Result

Theorem 2.1. *Let A, B, C, and D be any four distinct points on a given conic section from right to left, where if the section is either a circle or ellipse, then points are arranged such that no two tangent lines to those points are parallel.*
Let line AC intersect line BD at point E.
Let the tangent line through B meet the tangent line through C at point F.
Let line AB intersect line CD at point G.
Let the tangent line through A meet the tangent line through D at point H.
Let the tangent line through B meet line CD at point I.
Let line AB intersect the tangent line through C at point J.
Let line AB intersect the tangent line through D at point K.

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Let the tangent line through A intersect line CD at point L.
Let the tangent line through A intersect the tangent line through C at point M.
Let the tangent line through B intersect the tangent line through D at point N.
Let the tangent line through A intersect the tangent line through B at point O.
Let the tangent line through C intersect the tangent line through D at point P.
Then
1. *E, F, G and H are collinear,*
2. *E, I, and K are collinear,*
3. *E, J, and L are collinear,*
4. *G, M, and N are collinear,*
5. *E, O, and P are collinear, and*
6. *Lines AD, BC, IJ, KL, GMN, and EOP are either all concurrent or parallel.*

3. Parabolas

We will use cartesian coordinates. Without loss of generality, we take the parabola to be the equation given by $y = px^2$ for some $p \in \mathbb{R}$, since any parabola in the plane can be rotated and translated into a quadratic curve of this form.
A has coordinates given by (x_A, px_A^2).
B has coordinates given by (x_B, px_B^2).
C has coordinates given by (x_C, px_C^2).
D has coordinates given by (x_D, px_D^2).
The line through points A and B is given by the equation
$y = p(x(x_A + x_B) - x_A x_B)$.
The line through points C and D is given by the equation
$y = p(x(x_C + x_D) - x_C x_D)$.
The line through points A and C is given by the equation
$y = px_A^2 + p(x - x_A)(x_A + x_C)$.
The line through points B and D is given by the equation
$y = px_B^2 + p(x - x_B)(x_B + x_D)$.
The line through points A and D is given by the equation
$y = p(x(x_A + x_D) - x_A x_D)$.
The line through points B and C is given by the equation
$y = px_B^2 + p(x - x_B)(x_B + x_C)$.
The line tangent to A is given by the equation $y = px_A^2 + 2px_A(x - x_A)$.
The line tangent to B is given by the equation $y = px_B^2 + 2px_B(x - x_B)$.
The line tangent to C is given by the equation $y = px_C^2 + 2px_C(x - x_C)$.
The line tangent to D is given by the equation $y = px_D^2 + 2px_D(x - x_D)$.
The coordinates of E are given by
$x_E = \frac{x_A x_C - x_B x_D}{x_A + x_C - x_B - x_D}$ and $y_E = \frac{p(((x_B - x_A)x_C + x_A x_B)x_D - x_A x_B x_C)}{x_B + x_D - x_A - x_C}$.
The coordinates of F are given by $x_F = \frac{1}{2}(x_B + x_C)$ and $y_F = px_B x_C$.
The coordinates of G are given by
$x_G = \frac{x_A x_B - x_C x_D}{x_A + x_B - x_C - x_D}$ and $y_G = \frac{p(((x_A + x_B)x_C - x_A x_B)x_D - x_A x_B x_C)}{x_C + x_D - x_A - x_B}$.
The coordinates of H are given by $x_H = \frac{1}{2}(x_A + x_D)$ and $y_H = px_A x_D$.
The coordinates of I are given by $x_I = \frac{x_C x_D - x_B^2}{x_C + x_D - 2x_B}$ and $y_I = \frac{px_B(x_D(2x_C - x_B) - x_B x_C)}{x_C + x_D - 2x_B}$.
The coordinates of J are given by $x_J = \frac{x_A x_B - x_C^2}{x_A + x_B - 2x_C}$ and $y_J = \frac{px_C(x_C(x_A + x_B) - 2x_A x_B)}{2x_C - x_A - x_B}$.
The coordinates of K are given by $x_K = \frac{x_A x_B - x_D^2}{x_A + x_B - 2x_D}$ and $y_K = \frac{px_D(x_D(x_A + x_B) - 2x_A x_B)}{2x_D - x_A - x_B}$.
The coordinates of L are given by $x_L = \frac{x_C x_D - x_A^2}{x_C + x_D - 2x_A}$ and $y_L = \frac{px_A(x_D(2x_C - x_A) - x_A x_C)}{x_C + x_D - 2x_A}$.

The coordinates of M are given by $x_M = \frac{1}{2}(x_A + x_C)$ and $y_M = px_Ax_C$.
The coordinates of N are given by $x_N = \frac{1}{2}(x_B + x_D)$ and $y_N = px_Bx_D$.
The coordinates of O are given by $x_O = \frac{1}{2}(x_A + x_B)$ and $y_O = px_Ax_B$.
The coordinates of P are given by $x_P = \frac{1}{2}(x_C + x_D)$ and $y_P = px_Cx_D$.

1. The line passing through E and F is given by the equation
$y = y_E + (\frac{y_E - y_F}{x_E - x_F})(x - x_E)$.
The line passing through G and H is given by the equation
$y = y_G + (\frac{y_G - y_H}{x_G - x_H})(x - x_G)$.
We subtract one from the other and we get
$y_E + (\frac{y_E - y_F}{x_E - x_F})(x - x_E) - (y_G + (\frac{y_G - y_H}{x_G - x_H})(x - x_G))$
$= y_E - y_G - (\frac{y_E - y_F}{x_E - x_F})x_E + (\frac{y_G - y_H}{x_G - x_H})x_G + x(\frac{y_E - y_F}{x_E - x_F} - \frac{y_G - y_H}{x_G - x_H}) = 0$
since we calculate $\frac{y_E - y_F}{x_E - x_F} - \frac{y_G - y_H}{x_G - x_H}$
$= \frac{\frac{p(((x_B - x_A)x_C + x_Ax_B)x_D - x_Ax_Bx_C)}{x_B + x_D - x_A - x_C} - px_Bx_C}{\frac{x_Bx_D - x_Ax_C}{x_B + x_D - x_A - x_C} - \frac{1}{2}(x_B + x_C)} - \frac{\frac{p(((x_A + x_B)x_C - x_Ax_B)x_D - x_Ax_Bx_C)}{x_C + x_D - x_A - x_B} - px_Ax_D}{\frac{x_Ax_B - x_Cx_D}{x_A + x_B - x_C - x_D} - \frac{1}{2}(x_A + x_D)} = 0$
and $y_E - y_G - (\frac{y_E - y_F}{x_E - x_F})x_E + (\frac{y_G - y_H}{x_G - x_H})x_G$
$= (\frac{p(((x_B - x_A)x_C + x_Ax_B)x_D - x_Ax_Bx_C)}{x_B + x_D - x_A - x_C}) - (\frac{p(((x_A + x_B)x_C - x_Ax_B)x_D - x_Ax_Bx_C)}{x_C + x_D - x_A - x_B})$
$- (\frac{(\frac{p(((x_B - x_A)x_C + x_Ax_B)x_D - x_Ax_Bx_C)}{x_B + x_D - x_A - x_C}) - px_Bx_C}{(\frac{x_Bx_D - x_Ax_C}{x_B + x_D - x_A - x_C}) - \frac{1}{2}(x_B + x_C)})(\frac{x_Bx_D - x_Ax_C}{x_B + x_D - x_A - x_C})$
$+ (\frac{(\frac{p(((x_A + x_B)x_C - x_Ax_B)x_D - x_Ax_Bx_C)}{x_C + x_D - x_A - x_B}) - px_Ax_D}{(\frac{x_Ax_B - x_Cx_D}{x_A + x_B - x_C - x_D}) - \frac{1}{2}(x_A + x_D)})(\frac{x_Ax_B - x_Cx_D}{x_A + x_B - x_C - x_D}) = 0$ so E, F, G, and H are indeed collinear.

2. The line passing through E and I is given by the equation
$y = y_I + (\frac{y_E - y_I}{x_E - x_I})(x - x_I)$. We then verify that K lies on this line since
$y_I + (\frac{y_E - y_I}{x_E - x_I})(x_K - x_I) - y_K$
$= \frac{px_B(x_D(2x_C - x_B) - x_Bx_C)}{x_C + x_D - 2x_B} + (\frac{\frac{p(((x_B - x_A)x_C + x_Ax_B)x_D - x_Ax_Bx_C)}{x_B + x_D - x_A - x_C} - \frac{px_B(x_D(2x_C - x_B) - x_Bx_C)}{x_C + x_D - 2x_B}}{\frac{x_Ax_C - x_Bx_D}{x_A + x_C - x_B - x_D} - \frac{x_Cx_D - x_B^2}{x_C + x_D - 2x_B}})(\frac{x_Ax_B - x_D^2}{x_A + x_B - 2x_D} -$
$\frac{x_Cx_D - x_B^2}{x_C + x_D - 2x_B}) - \frac{px_D(x_D(x_A + x_B) - 2x_Ax_B)}{2x_D - x_A - x_B} = 0$ so E, I, and K are indeed collinear.

3. The line passing through E and J is given by the equation
$y = y_J + (\frac{y_E - y_J}{x_E - x_J})(x - x_J)$. We then verify that L lies on this line since
$y_J + (\frac{y_E - y_J}{x_E - x_J})(x_L - x_J) - y_L$
$= \frac{px_C(x_C(x_A + x_B) - 2x_Ax_B)}{2x_C - x_A - x_B} + (\frac{\frac{p(((x_B - x_A)x_C + x_Ax_B)x_D - x_Ax_Bx_C)}{x_B + x_D - x_A - x_C} - \frac{px_C(x_C(x_A + x_B) - 2x_Ax_B)}{2x_C - x_A - x_B}}{\frac{x_Ax_C - x_Bx_D}{x_A + x_C - x_B - x_D} - \frac{x_Ax_B - x_C^2}{x_A + x_B - 2x_C}})(\frac{x_Cx_D - x_A^2}{x_C + x_D - 2x_A} -$
$\frac{x_Ax_B - x_C^2}{x_A + x_B - 2x_C}) - \frac{px_A(x_D(2x_C - x_A) - x_Ax_C)}{x_C + x_D - 2x_A} = 0$ so E, J, and L are indeed collinear.

4. The line passing through G and M is given by the equation
$y = y_M + (\frac{y_G - y_M}{x_G - x_M})(x - x_M)$. We then verify that N lies on this line since
$y_M + (\frac{y_G - y_M}{x_G - x_M})(x_N - x_M) - y_N = px_Ax_C + (\frac{\frac{p(((x_A + x_B)x_C - x_Ax_B)x_D - x_Ax_Bx_C)}{x_C + x_D - x_A - x_B} - px_Ax_C}{\frac{x_Ax_B - x_Cx_D}{x_A + x_B - x_C - x_D} - \frac{1}{2}(x_A + x_C)})(\frac{1}{2}(x_B +$
$x_D) - \frac{1}{2}(x_A + x_C)) - px_Bx_D = 0$ so G, M, and N are indeed collinear.

5. The line passing through E and O is given by the equation $y = y_O + (\frac{y_E - y_O}{x_E - x_O})(x - x_O)$. We then verify that P lies on this line since $y_O + (\frac{y_E - y_O}{x_E - x_O})(x_P - x_O) - y_P = px_Ax_B + (\frac{\frac{p(((x_B - x_A)x_C + x_Ax_B)x_D - x_Ax_Bx_C)}{x_B + x_D - x_A - x_C} - px_Ax_B}{\frac{x_Ax_C - x_Bx_D}{x_A + x_C - x_B - x_D} - \frac{1}{2}(x_A + x_B)})(\frac{1}{2}(x_C + x_D) - \frac{1}{2}(x_A + x_B)) - px_Cx_D = 0$ so E, O, and P are indeed collinear.

6. We see easily that line AD and line BC intersect at the point $(\frac{x_Ax_D - x_Bx_C}{x_A + x_D - x_B - x_C}, \frac{-p(((x_B - x_A)x_C - x_Ax_B)x_D + x_Ax_Bx_C)}{x_A + x_D - x_B - x_C})$.
The line passing through I and J is given by the equation $y = p(x(x_I + x_J) - x_Ix_J)$.
The line passing through K and L is given by the equation $y = p(x(x_K + x_L) - x_Kx_L)$.
If these two lines intersect at a point Q, then the coordinates of Q are given by $x_Q = \frac{x_Ax_D - x_Bx_C}{x_A + x_D - x_B - x_C}$ and $y_Q = \frac{-p(((x_B - x_A)x_C - x_Ax_B)x_D + x_Ax_Bx_C)}{x_A + x_D - x_B - x_C}$.
The line passing through G, M, and N is given by the equation $y = y_M + (\frac{y_G - y_M}{x_G - x_M})(x - x_M)$.
The line passing through E, O, and P is given by the equation $y = y_O + (\frac{y_E - y_O}{x_E - x_O})(x - x_O)$
If these two lines intersect at a point Q', then the coordinates of Q' are given by $x_{Q'} = \frac{x_Ax_D - x_Bx_C}{x_A + x_D - x_B - x_C}$ and $y_{Q'} = \frac{-p(((x_B - x_A)x_C - x_Ax_B)x_D + x_Ax_Bx_C)}{x_A + x_D - x_B - x_C}$ so $Q = Q'$.
Now we note that if two of these lines are not concurrent, then none of them are concurrent, and so they all would have to be parallel.

4. Hyperbolas

We will use cartesian coordinates. Without loss of generality, we take the hyperbola to be the equation given by $y = q\sqrt{(\frac{x}{p})^2 + 1}$ for some positive real p, q since any hyperbola in the plane can be rotated and translated into a curve of this form.
A has coordinates given by $(x_A, q\sqrt{(\frac{x_A}{p})^2 + 1})$.
B has coordinates given by $(x_B, q\sqrt{(\frac{x_B}{p})^2 + 1})$.
C has coordinates given by $(x_C, q\sqrt{(\frac{x_C}{p})^2 + 1})$.
D has coordinates given by $(x_D, q\sqrt{(\frac{x_D}{p})^2 + 1})$.
The line passing through points A and B is given by the equation $y = \frac{q((x - x_A)\sqrt{x_B^2 + p^2} - (x - x_B)\sqrt{x_A^2 + p^2})}{p(x_B - x_A)}$.
The line passing through points C and D is given by the equation $y = \frac{q((x - x_C)\sqrt{x_D^2 + p^2} - (x - x_D)\sqrt{x_C^2 + p^2})}{p(x_D - x_C)}$.
The line passing through points A and C is given by the equation $y = \frac{q((x - x_A)\sqrt{x_C^2 + p^2} - (x - x_C)\sqrt{x_A^2 + p^2})}{p(x_C - x_A)}$.
The line passing through points B and D is given by the equation $y = \frac{q((x - x_B)\sqrt{x_D^2 + p^2} - (x - x_D)\sqrt{x_B^2 + p^2})}{p(x_D - x_B)}$.
The line passing through points A and D is given by the equation $y = \frac{q((x - x_A)\sqrt{x_D^2 + p^2} - (x - x_D)\sqrt{x_A^2 + p^2})}{p(x_D - x_A)}$.
The line passing through points B and C is given by the equation

$y = \frac{q((x-x_B)\sqrt{x_C^2+p^2}-(x-x_C)\sqrt{x_B^2+p^2})}{p(x_C-x_B)}$.

The line tangent to A is given by the equation $y = \frac{q(xx_A+p^2)}{p\sqrt{x_A^2+p^2}}$.

The line tangent to B is given by the equation $y = \frac{q(xx_B+p^2)}{p\sqrt{x_B^2+p^2}}$.

The line tangent to C is given by the equation $y = \frac{q(xx_C+p^2)}{p\sqrt{x_C^2+p^2}}$.

The line tangent to D is given by the equation $y = \frac{q(xx_D+p^2)}{p\sqrt{x_D^2+p^2}}$.

The coordinates of E are given by

$x_E = \frac{(x_C-x_A)(x_B\sqrt{x_D^2+p^2}-x_D\sqrt{x_B^2+p^2})+(x_D-x_B)(x_C\sqrt{x_A^2+p^2}-x_A\sqrt{x_C^2+p^2})}{(x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2})}$

and

$y_E = \frac{q(((x_B-x_A)\sqrt{x_C^2+p^2}+(x_C-x_B)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_C)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_D)\sqrt{(x_B^2+p^2)(x_C^2+p^2)})}{p((x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2}))}$.

The coordinates of F are given by

$x_F = \frac{x_Bx_C-p^2+\sqrt{(x_B^2+p^2)(x_C^2+p^2)}}{x_B+x_C}$ and $y_F = \frac{q(x_B\sqrt{x_C^2+p^2}+x_C\sqrt{x_B^2+p^2})}{p(x_B+x_C)}$.

The coordinates of G are given by

$x_G = \frac{(x_B-x_A)(x_C\sqrt{x_D^2+p^2}-x_D\sqrt{x_C^2+p^2})+(x_D-x_C)(x_B\sqrt{x_A^2+p^2}-x_A\sqrt{x_B^2+p^2})}{(x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})}$ and

$y_G = \frac{q(((x_C-x_A)\sqrt{x_B^2+p^2}+(x_B-x_C)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{(x_A^2+p^2)(x_C^2+p^2)}+(x_A-x_D)\sqrt{(x_C^2+p^2)(x_B^2+p^2)})}{p((x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2}))}$.

The coordinates of H are given by

$x_H = \frac{x_Ax_D-p^2+\sqrt{(x_A^2+p^2)(x_D^2+p^2)}}{x_A+x_D}$ and $y_H = \frac{q(x_A\sqrt{x_D^2+p^2}+x_D\sqrt{x_A^2+p^2})}{p(x_A+x_D)}$.

The coordinates of I are given by

$x_I = \frac{p^2(x_C-x_D)+x_D\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B(x_D-x_C)+\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}$ and

$y_I = \frac{q(x_Bx_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Bx_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_B(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_B^2+p^2})}$.

The coordinates of J are given by

$x_J = \frac{p^2(x_A-x_B)+x_B\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_C^2+p^2)(x_B^2+p^2)}}{x_C(x_B-x_A)+\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-\sqrt{(x_C^2+p^2)(x_B^2+p^2)}}$ and

$y_J = \frac{q(x_Cx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Cx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_C(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2})}$.

The coordinates of K are given by

$x_K = \frac{p^2(x_A-x_B)+x_B\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_D^2+p^2)(x_B^2+p^2)}}{x_D(x_B-x_A)+\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-\sqrt{(x_D^2+p^2)(x_B^2+p^2)}}$ and

$y_K = \frac{q(x_Dx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Dx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_D(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2})}$.

The coordinates of L are given by

$x_L = \frac{p^2(x_C-x_D)+x_D\sqrt{(x_A^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_A^2+p^2)(x_D^2+p^2)}}{x_A(x_D-x_C)+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}-\sqrt{(x_A^2+p^2)(x_D^2+p^2)}}$ and

$y_L = \frac{q(x_Ax_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Ax_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_A(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_A^2+p^2})}$.

The coordinates of M are given by

$x_M = \frac{x_Ax_C-p^2+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}}{x_A+x_C}$ and $y_M = \frac{q(x_A\sqrt{x_C^2+p^2}+x_C\sqrt{x_A^2+p^2})}{p(x_A+x_C)}$.

The coordinates of N are given by

$x_N = \frac{x_Bx_D-p^2+\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B+x_D}$ and $y_N = \frac{q(x_B\sqrt{x_D^2+p^2}+x_D\sqrt{x_B^2+p^2})}{p(x_B+x_D)}$.

The coordinates of O are given by

$x_O = \frac{x_A x_B - p^2 + \sqrt{(x_A^2+p^2)(x_B^2+p^2)}}{x_A+x_B}$ and $y_O = \frac{q(x_A\sqrt{x_B^2+p^2}+x_B\sqrt{x_A^2+p^2})}{p(x_A+x_B)}$.
The coordinates of P are given by
$x_P = \frac{x_C x_D - p^2 + \sqrt{(x_C^2+p^2)(x_D^2+p^2)}}{x_C+x_D}$ and $y_P = \frac{q(x_C\sqrt{x_D^2+p^2}+x_D\sqrt{x_C^2+p^2})}{p(x_C+x_D)}$.

1. The line passing through E and F is given by the equation
$y = y_E + (\frac{y_E-y_F}{x_E-x_F})(x - x_E)$.
The line passing through G and H is given by the equation
$y = y_G + (\frac{y_G-y_H}{x_G-x_H})(x - x_G)$.
We subtract one from the other and we get
$y_E + (\frac{y_E-y_F}{x_E-x_F})(x - x_E) - (y_G + (\frac{y_G-y_H}{x_G-x_H})(x - x_G))$
$= y_E - y_G - (\frac{y_E-y_F}{x_E-x_F})x_E + (\frac{y_G-y_H}{x_G-x_H})x_G + x(\frac{y_E-y_F}{x_E-x_F} - \frac{y_G-y_H}{x_G-x_H}) = 0$
since we calculate
$(y_E - y_F)(x_G - x_H) - (y_G - y_H)(x_E - x_F)$
$= ((\frac{q(((x_B-x_A)\sqrt{x_C^2+p^2}+(x_C-x_B)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_C)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_D)\sqrt{(x_B^2+p^2)(x_C^2+p^2)})}{p((x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2}))})$
$-(\frac{q(x_B\sqrt{x_C^2+p^2}+x_C\sqrt{x_B^2+p^2})}{p(x_B+x_C)}))((\frac{(x_B-x_A)(x_C\sqrt{x_D^2+p^2}-x_D\sqrt{x_C^2+p^2})+(x_D-x_C)(x_B\sqrt{x_A^2+p^2}-x_A\sqrt{x_B^2+p^2})}{(x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})})$
$-(\frac{x_A x_D - p^2 + \sqrt{(x_A^2+p^2)(x_D^2+p^2)}}{x_A+x_D}))$
$-((\frac{q(((x_C-x_A)\sqrt{x_B^2+p^2}+(x_B-x_C)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{(x_A^2+p^2)(x_C^2+p^2)}+(x_A-x_D)\sqrt{(x_C^2+p^2)(x_B^2+p^2)})}{p((x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2}))})$
$-(\frac{q(x_A\sqrt{x_D^2+p^2}+x_D\sqrt{x_A^2+p^2})}{p(x_A+x_D)}))((\frac{(x_C-x_A)(x_B\sqrt{x_D^2+p^2}-x_D\sqrt{x_B^2+p^2})+(x_D-x_B)(x_C\sqrt{x_A^2+p^2}-x_A\sqrt{x_C^2+p^2})}{(x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2})})$
$-(\frac{x_B x_C - p^2 + \sqrt{(x_B^2+p^2)(x_C^2+p^2)}}{x_B+x_C})) = 0$
and using this we find that
$(y_E - y_G)(x_E - x_F)(x_G - x_H) + x_G(x_E - x_F)(y_G - y_H) - x_E(y_E - y_F)(x_G - x_H)$
$= (y_E - y_G)(x_E - x_F)(x_G - x_H) + x_G(y_E - y_F)(x_G - x_H) - x_E(y_E - y_F)(x_G - x_H)$
$= (x_G - x_H)[(y_E - y_G)(x_E - x_F) + (x_G - x_E)(y_E - y_F)]$
$= (x_G - x_H)[x_F y_G + x_G y_E + x_E y_F - x_F y_E - x_E y_G - x_G y_F]$ and thus we calculate
that
$x_F y_G + x_G y_E + x_E y_F - x_F y_E - x_E y_G - x_G y_F$
$= (\frac{x_B x_C - p^2 + \sqrt{(x_B^2+p^2)(x_C^2+p^2)}}{x_B+x_C})$
$*(\frac{q(((x_C-x_A)\sqrt{x_B^2+p^2}+(x_B-x_C)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{(x_A^2+p^2)(x_C^2+p^2)}+(x_A-x_D)\sqrt{(x_C^2+p^2)(x_B^2+p^2)})}{p((x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2}))})$
$+(\frac{(x_B-x_A)(x_C\sqrt{x_D^2+p^2}-x_D\sqrt{x_C^2+p^2})+(x_D-x_C)(x_B\sqrt{x_A^2+p^2}-x_A\sqrt{x_B^2+p^2})}{(x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})})$
$*(\frac{q(((x_B-x_A)\sqrt{x_C^2+p^2}+(x_C-x_B)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_C)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_D)\sqrt{(x_B^2+p^2)(x_C^2+p^2)})}{p((x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2}))})$
$+(\frac{(x_C-x_A)(x_B\sqrt{x_D^2+p^2}-x_D\sqrt{x_B^2+p^2})+(x_D-x_B)(x_C\sqrt{x_A^2+p^2}-x_A\sqrt{x_C^2+p^2})}{(x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2})})$
$*(\frac{q(x_B\sqrt{x_C^2+p^2}+x_C\sqrt{x_B^2+p^2})}{p(x_B+x_C)})$
$-(\frac{x_B x_C - p^2 + \sqrt{(x_B^2+p^2)(x_C^2+p^2)}}{x_B+x_C})$
$*(\frac{q(((x_B-x_A)\sqrt{x_C^2+p^2}+(x_C-x_B)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_C)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_D)\sqrt{(x_B^2+p^2)(x_C^2+p^2)})}{p((x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2}))})$
$-(\frac{(x_C-x_A)(x_B\sqrt{x_D^2+p^2}-x_D\sqrt{x_B^2+p^2})+(x_D-x_B)(x_C\sqrt{x_A^2+p^2}-x_A\sqrt{x_C^2+p^2})}{(x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2})})$

$*(\frac{q(((x_C-x_A)\sqrt{x_B^2+p^2}+(x_B-x_C)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{(x_A^2+p^2)(x_C^2+p^2)}+(x_A-x_D)\sqrt{(x_C^2+p^2)(x_B^2+p^2)})}{p((x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2}))})$
$-(\frac{(x_B-x_A)(x_C\sqrt{x_D^2+p^2}-x_D\sqrt{x_C^2+p^2})+(x_D-x_C)(x_B\sqrt{x_A^2+p^2}-x_A\sqrt{x_B^2+p^2})}{(x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})})(\frac{q(x_B\sqrt{x_C^2+p^2}+x_C\sqrt{x_B^2+p^2})}{p(x_B+x_C)})$
$= 0$ so E, F, G, and H are indeed collinear.

2. The line passing through E and I is given by the equation
$y = y_I + (\frac{y_E-y_I}{x_E-x_I})(x-x_I)$.
The line passing through I and K is given by the equation
$y = y_I + (\frac{y_K-y_I}{x_K-x_I})(x-x_I)$.
We subtract one from the other and we get
$y_I + (\frac{y_E-y_I}{x_E-x_I})(x-x_I) - (y_I + (\frac{y_K-y_I}{x_K-x_I})(x-x_I))$
$= x_I(\frac{y_K-y_I}{x_K-x_I} - \frac{y_E-y_I}{x_E-x_I}) + x(\frac{y_K-y_I}{x_K-x_I} - \frac{y_E-y_I}{x_E-x_I}) = 0$ since we calculate
$(y_K-y_I)(x_E-x_I) - (y_E-y_I)(x_K-x_I)$
$= ((\frac{q(x_Dx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Dx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_D(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2})})$
$-(\frac{q(x_Bx_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Bx_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_B(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_B^2+p^2})}))$
$*((\frac{(x_C-x_A)(x_B\sqrt{x_D^2+p^2}-x_D\sqrt{x_B^2+p^2})+(x_D-x_B)(x_C\sqrt{x_A^2+p^2}-x_A\sqrt{x_C^2+p^2})}{(x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2})})$
$-(\frac{p^2(x_C-x_D)+x_D\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B(x_D-x_C)+\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}))$
$-((\frac{q(((x_B-x_A)\sqrt{x_C^2+p^2}+(x_C-x_B)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_C)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_D)\sqrt{(x_B^2+p^2)(x_C^2+p^2)})}{p((x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2}))})$
$-(\frac{q(x_Bx_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Bx_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_B(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_B^2+p^2})}))$
$*((\frac{p^2(x_A-x_B)+x_B\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_D^2+p^2)(x_B^2+p^2)}}{x_D(x_B-x_A)+\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-\sqrt{(x_D^2+p^2)(x_B^2+p^2)}})$
$-(\frac{p^2(x_C-x_D)+x_D\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B(x_D-x_C)+\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-\sqrt{(x_B^2+p^2)(x_D^2+p^2)}})) = 0$ so E, I, and K are indeed collinear.

3. The line passing through E and J is given by the equation
$y = y_J + (\frac{y_E-y_J}{x_E-x_J})(x-x_J)$.
The line passing through J and L is given by the equation
$y = y_J + (\frac{y_L-y_J}{x_L-x_J})(x-x_J)$.
We subtract one from the other and we get
$y_J + (\frac{y_E-y_J}{x_E-x_J})(x-x_J) - (y_J + (\frac{y_L-y_J}{x_L-x_J})(x-x_J))$
$= x_J(\frac{y_L-y_J}{x_L-x_J} - \frac{y_E-y_J}{x_E-x_J}) + x(\frac{y_L-y_J}{x_L-x_J} - \frac{y_E-y_J}{x_E-x_J}) = 0$ since we calculate
$(y_L-y_J)(x_E-x_J) - (y_E-y_J)(x_L-x_J)$
$= ((\frac{q(x_Ax_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Ax_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_A(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_A^2+p^2})})$
$-(\frac{q(x_Cx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Cx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_C(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2})}))$
$*((\frac{(x_C-x_A)(x_B\sqrt{x_D^2+p^2}-x_D\sqrt{x_B^2+p^2})+(x_D-x_B)(x_C\sqrt{x_A^2+p^2}-x_A\sqrt{x_C^2+p^2})}{(x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2})})$
$-(\frac{p^2(x_A-x_B)+x_B\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_C^2+p^2)(x_B^2+p^2)}}{x_C(x_B-x_A)+\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-\sqrt{(x_C^2+p^2)(x_B^2+p^2)}}))$

$-((\frac{q(((x_B-x_A)\sqrt{x_C^2+p^2}+(x_C-x_B)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_C)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_D)\sqrt{(x_B^2+p^2)(x_C^2+p^2)})}{p((x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2}))}$
$-(\frac{q(x_Cx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Cx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_C(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2})}))$
$*((\frac{p^2(x_C-x_D)+x_D\sqrt{(x_A^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_A^2+p^2)(x_D^2+p^2)}}{x_A(x_D-x_C)+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}-\sqrt{(x_A^2+p^2)(x_D^2+p^2)}})$
$-(\frac{p^2(x_A-x_B)+x_B\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_C^2+p^2)(x_B^2+p^2)}}{x_C(x_B-x_A)+\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-\sqrt{(x_C^2+p^2)(x_B^2+p^2)}}))=0$ so E, J, and L are indeed collinear.

4. The line passing through G and M is given by the equation
$y=y_M+(\frac{y_G-y_M}{x_G-x_M})(x-x_M)$.
The line passing through M and N is given by the equation
$y=y_M+(\frac{y_N-y_M}{x_N-x_M})(x-x_M)$.
We subtract one from the other and we get
$y_M+(\frac{y_G-y_M}{x_G-x_M})(x-x_M)-(y_M+(\frac{y_N-y_M}{x_N-x_M})(x-x_M))$
$=x_M(\frac{y_N-y_M}{x_N-x_M}-\frac{y_G-y_M}{x_G-x_M})+x(\frac{y_N-y_M}{x_N-x_M}-\frac{y_G-y_M}{x_G-x_M})=0$ since we calculate
$(y_N-y_M)(x_G-x_M)-(y_G-y_M)(x_N-x_M)$
$=((\frac{q(x_B\sqrt{x_D^2+p^2}+x_D\sqrt{x_B^2+p^2})}{p(x_B+x_D)})-(\frac{q(x_A\sqrt{x_C^2+p^2}+x_C\sqrt{x_A^2+p^2})}{p(x_A+x_C)}))$
$*((\frac{(x_B-x_A)(x_C\sqrt{x_D^2+p^2}-x_D\sqrt{x_C^2+p^2})+(x_D-x_C)(x_B\sqrt{x_A^2+p^2}-x_A\sqrt{x_B^2+p^2})}{(x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})})-(\frac{x_Ax_C-p^2+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}}{x_A+x_C}))$
$-((\frac{q(((x_C-x_A)\sqrt{x_B^2+p^2}+(x_B-x_C)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{(x_A^2+p^2)(x_C^2+p^2)}+(x_A-x_D)\sqrt{(x_C^2+p^2)(x_B^2+p^2)})}{p((x_B-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})-(x_D-x_C)(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2}))})$
$-(\frac{q(x_A\sqrt{x_C^2+p^2}+x_C\sqrt{x_A^2+p^2})}{p(x_A+x_C)}))*((\frac{x_Bx_D-p^2+\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B+x_D})-(\frac{x_Ax_C-p^2+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}}{x_A+x_C}))$
$=0$ so G, M, and N are indeed collinear.

5. The line passing through E and O is given by the equation
$y=y_O+(\frac{y_E-y_O}{x_E-x_O})(x-x_O)$.
The line passing through O and P is given by the equation
$y=y_O+(\frac{y_P-y_O}{x_P-x_O})(x-x_O)$.
We subtract one from the other and we get
$y_O+(\frac{y_E-y_O}{x_E-x_O})(x-x_O)-(y_O+(\frac{y_P-y_O}{x_P-x_O})(x-x_O))$
$=x_O(\frac{y_P-y_O}{x_P-x_O}-\frac{y_E-y_O}{x_E-x_O})+x(\frac{y_P-y_O}{x_P-x_O}-\frac{y_E-y_O}{x_E-x_O})=0$
since we calculate
$(y_P-y_O)(x_E-x_O)-(y_E-y_O)(x_P-x_O)$
$=((\frac{q(x_C\sqrt{x_D^2+p^2}+x_D\sqrt{x_C^2+p^2})}{p(x_C+x_D)})-(\frac{q(x_A\sqrt{x_B^2+p^2}+x_B\sqrt{x_A^2+p^2})}{p(x_A+x_B)}))$
$*((\frac{(x_C-x_A)(x_B\sqrt{x_D^2+p^2}-x_D\sqrt{x_B^2+p^2})+(x_D-x_B)(x_C\sqrt{x_A^2+p^2}-x_A\sqrt{x_C^2+p^2})}{(x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2})})-(\frac{x_Ax_B-p^2+\sqrt{(x_A^2+p^2)(x_B^2+p^2)}}{x_A+x_B}))$
$-((\frac{q(((x_B-x_A)\sqrt{x_C^2+p^2}+(x_C-x_B)\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2}+(x_D-x_C)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_D)\sqrt{(x_B^2+p^2)(x_C^2+p^2)})}{p((x_C-x_A)(\sqrt{x_D^2+p^2}-\sqrt{x_B^2+p^2})-(x_D-x_B)(\sqrt{x_C^2+p^2}-\sqrt{x_A^2+p^2}))})$
$-(\frac{q(x_A\sqrt{x_B^2+p^2}+x_B\sqrt{x_A^2+p^2})}{p(x_A+x_B)}))*((\frac{x_Cx_D-p^2+\sqrt{(x_C^2+p^2)(x_D^2+p^2)}}{x_C+x_D})-(\frac{x_Ax_B-p^2+\sqrt{(x_A^2+p^2)(x_B^2+p^2)}}{x_A+x_B}))$
$=0$ so E, O, and P are indeed collinear.

6. Line AD is given by the equation $y=\frac{q((x-x_A)\sqrt{x_D^2+p^2}-(x-x_D)\sqrt{x_A^2+p^2})}{p(x_D-x_A)}$.
Line BC is given by the equation $y=\frac{q((x-x_C)\sqrt{x_B^2+p^2}-(x-x_B)\sqrt{x_C^2+p^2})}{p(x_B-x_C)}$.

We see that if these intersect at point Q, then Q has coordinates given by
$x_Q = \frac{(x_D-x_A)(x_B\sqrt{x_C^2+p^2}-x_C\sqrt{x_B^2+p^2})+(x_C-x_B)(x_D\sqrt{x_A^2+p^2}-x_A\sqrt{x_D^2+p^2})}{(x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2})}$ and
$y_Q = \frac{q(((x_B-x_A)\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2}+(x_C-x_D)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_C)\sqrt{(x_B^2+p^2)(x_D^2+p^2)})}{p((x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2}))}$.
We verify that Q lies on line IJ since
$(\frac{q(x_Bx_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Bx_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_B(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_B^2+p^2})}$
$+ ((\frac{(x_D-x_A)(x_B\sqrt{x_C^2+p^2}-x_C\sqrt{x_B^2+p^2})+(x_C-x_B)(x_D\sqrt{x_A^2+p^2}-x_A\sqrt{x_D^2+p^2})}{(x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2})}$
$- (\frac{p^2(x_C-x_D)+x_D\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B(x_D-x_C)+\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}))$
$*(\frac{(\frac{q(x_Cx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Cx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_C(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2})}-(\frac{q(x_Bx_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Bx_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_B(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_B^2+p^2})})}{(\frac{p^2(x_A-x_B)+x_B\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_C^2+p^2)(x_B^2+p^2)}}{x_C(x_B-x_A)+\sqrt{(x_C^2+p^2)(x_A^2+p^2)}-\sqrt{(x_C^2+p^2)(x_B^2+p^2)}})-(\frac{p^2(x_C-x_D)+x_D\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B(x_D-x_C)+\sqrt{(x_B^2+p^2)(x_C^2+p^2)}-\sqrt{(x_B^2+p^2)(x_D^2+p^2)}})}-$
$(\frac{q(((x_B-x_A)\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2}+(x_C-x_D)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_C)\sqrt{(x_B^2+p^2)(x_D^2+p^2)})}{p((x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2}))}) =$
0.
We verify that Q lies on line KL since
$(\frac{q(x_Dx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Dx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_D(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2})}$
$+ ((\frac{(x_D-x_A)(x_B\sqrt{x_C^2+p^2}-x_C\sqrt{x_B^2+p^2})+(x_C-x_B)(x_D\sqrt{x_A^2+p^2}-x_A\sqrt{x_D^2+p^2})}{(x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2})}$
$- (\frac{p^2(x_A-x_B)+x_B\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_D^2+p^2)(x_B^2+p^2)}}{x_D(x_B-x_A)+\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-\sqrt{(x_D^2+p^2)(x_B^2+p^2)}}))$
$*(\frac{(\frac{q(x_Ax_C\sqrt{x_D^2+p^2}+p^2\sqrt{x_D^2+p^2}-x_Ax_D\sqrt{x_C^2+p^2}-p^2\sqrt{x_C^2+p^2})}{p(x_A(x_C-x_D)+(\sqrt{x_D^2+p^2}-\sqrt{x_C^2+p^2})\sqrt{x_A^2+p^2})}-(\frac{q(x_Dx_A\sqrt{x_B^2+p^2}+p^2\sqrt{x_B^2+p^2}-x_Dx_B\sqrt{x_A^2+p^2}-p^2\sqrt{x_A^2+p^2})}{p(x_D(x_A-x_B)+(\sqrt{x_B^2+p^2}-\sqrt{x_A^2+p^2})\sqrt{x_D^2+p^2})})}{(\frac{p^2(x_C-x_D)+x_D\sqrt{(x_A^2+p^2)(x_C^2+p^2)}-x_C\sqrt{(x_A^2+p^2)(x_D^2+p^2)}}{x_A(x_D-x_C)+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}-\sqrt{(x_A^2+p^2)(x_D^2+p^2)}})-(\frac{p^2(x_A-x_B)+x_B\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-x_A\sqrt{(x_D^2+p^2)(x_B^2+p^2)}}{x_D(x_B-x_A)+\sqrt{(x_D^2+p^2)(x_A^2+p^2)}-\sqrt{(x_D^2+p^2)(x_B^2+p^2)}})}-$
$(\frac{q(((x_B-x_A)\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2}+(x_C-x_D)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_C)\sqrt{(x_B^2+p^2)(x_D^2+p^2)})}{p((x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2}))}) =$
0.
We verify that Q lies on line GMN since
$(\frac{q(x_A\sqrt{x_C^2+p^2}+x_C\sqrt{x_A^2+p^2})}{p(x_A+x_C)})+((\frac{(x_D-x_A)(x_B\sqrt{x_C^2+p^2}-x_C\sqrt{x_B^2+p^2})+(x_C-x_B)(x_D\sqrt{x_A^2+p^2}-x_A\sqrt{x_D^2+p^2})}{(x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2})}-$
$(\frac{x_Ax_C-p^2+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}}{x_A+x_C}))(\frac{(\frac{q(x_B\sqrt{x_D^2+p^2}+x_D\sqrt{x_B^2+p^2})}{p(x_B+x_D)})-(\frac{q(x_A\sqrt{x_C^2+p^2}+x_C\sqrt{x_A^2+p^2})}{p(x_A+x_C)})}{(\frac{x_Bx_D-p^2+\sqrt{(x_B^2+p^2)(x_D^2+p^2)}}{x_B+x_D})-(\frac{x_Ax_C-p^2+\sqrt{(x_A^2+p^2)(x_C^2+p^2)}}{x_A+x_C})}$
$-(\frac{q(((x_B-x_A)\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2}+(x_C-x_D)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_C)\sqrt{(x_B^2+p^2)(x_D^2+p^2)})}{p((x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2}))})$
$= 0.$
We verify that Q lies on line EOP since
$(\frac{q(x_A\sqrt{x_B^2+p^2}+x_B\sqrt{x_A^2+p^2})}{p(x_A+x_B)})+((\frac{(x_D-x_A)(x_B\sqrt{x_C^2+p^2}-x_C\sqrt{x_B^2+p^2})+(x_C-x_B)(x_D\sqrt{x_A^2+p^2}-x_A\sqrt{x_D^2+p^2})}{(x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2})}-$
$(\frac{x_Ax_B-p^2+\sqrt{(x_A^2+p^2)(x_B^2+p^2)}}{x_A+x_B}))(\frac{(\frac{q(x_C\sqrt{x_D^2+p^2}+x_D\sqrt{x_C^2+p^2})}{p(x_C+x_D)})-(\frac{q(x_A\sqrt{x_B^2+p^2}+x_B\sqrt{x_A^2+p^2})}{p(x_A+x_B)})}{(\frac{x_Cx_D-p^2+\sqrt{(x_C^2+p^2)(x_D^2+p^2)}}{x_C+x_D})-(\frac{x_Ax_B-p^2+\sqrt{(x_A^2+p^2)(x_B^2+p^2)}}{x_A+x_B})}$
$-(\frac{q(((x_B-x_A)\sqrt{x_D^2+p^2}+(x_D-x_B)\sqrt{x_A^2+p^2})\sqrt{x_C^2+p^2}+(x_C-x_D)\sqrt{(x_A^2+p^2)(x_B^2+p^2)}+(x_A-x_C)\sqrt{(x_B^2+p^2)(x_D^2+p^2)})}{p((x_D-x_A)(\sqrt{x_C^2+p^2}-\sqrt{x_B^2+p^2})-(x_C-x_B)(\sqrt{x_D^2+p^2}-\sqrt{x_A^2+p^2}))})$

$= 0$ as desired.

We notice that if we're considering the second hyperbola mirrored by the first, then the above propositions will still hold because if any of the signs of the y values of A, B, C, or D are switched, then the above calculations will still hold.

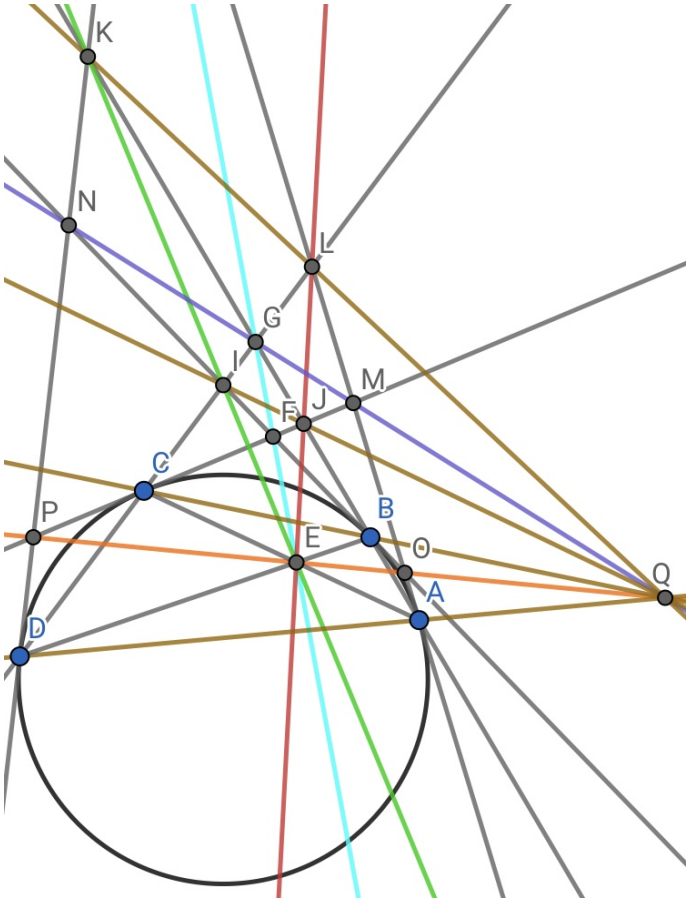

FIGURE 1. When the conic section is a circle, E, F, G, and H are collinear, the triples of points $(E, I, K), (E, J, L), (G, M, N)$, and (E, O, P) are each collinear, and lines AD, BC, IJ, KL, GMN, and EOP are concurrent at point Q.

REFERENCES

[1] Warren, B.L., "A Collection of Collinear Points", *International Journal of Computer Discovered Mathematics*, Vol. 8 (2023), paper 10, pp. 65-69.

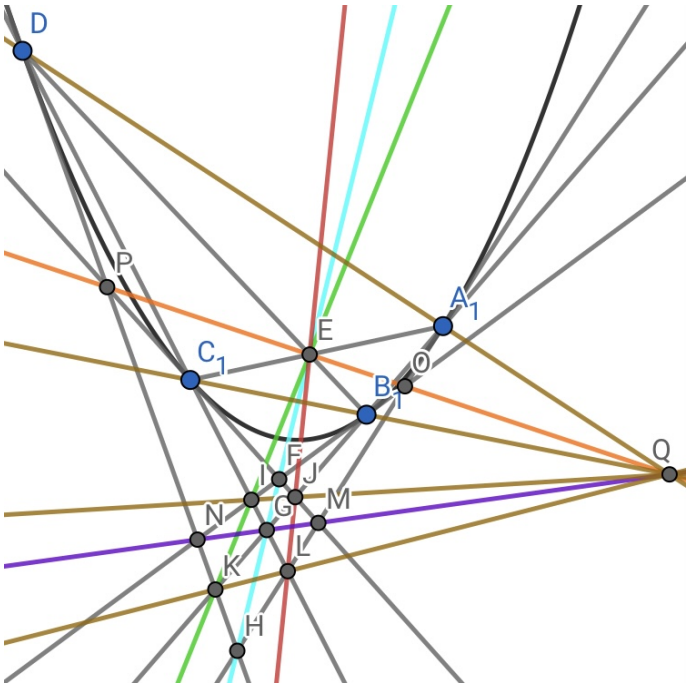

Figure 2. When the conic section is a parabola, E, F, G, and H are collinear, the triples of points $(E, I, K), (E, J, L), (G, M, N)$, and (E, O, P) are each collinear, and lines AD, BC, IJ, KL, GMN, and EOP are concurrent at point Q.

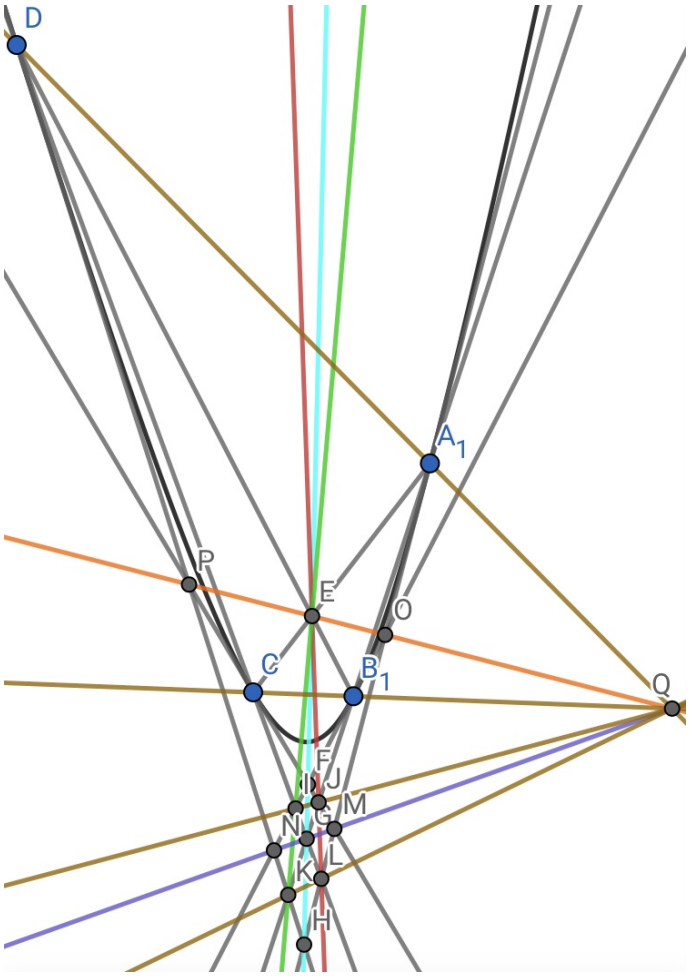

FIGURE 3. When the conic section is a hyperbola, E, F, G, and H are collinear, the triples of points $(E, I, K), (E, J, L), (G, M, N)$, and (E, O, P) are each collinear, and lines AD, BC, IJ, KL, GMN, and EOP are concurrent at point Q.