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More on the Circles Which are Centered at Vertices of a Triangle and Passing Through the Rest of the Vertices

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1. Introduction

In [1], a handful of properties were discussed regarding the circles which are centered at the vertices of a given triangle, and which pass through the other vertices. Here, we demonstrate a small but related result regarding when four points are collinear in reference to this same construction.

2. Preliminaries

Let A, B, and C denote the vertices of a given triangle.
Let $\mathcal{O}Ab$ denote the circle centered at A which passes through B.
Let $\mathcal{O}Ba$ denote the circle centered at B which passes through A.
Let $\mathcal{O}Ac$ denote the circle centered at A which passes through C.
Let $\mathcal{O}Ca$ denote the circle centered at C which passes through A.
Let $\mathcal{O}Bc$ denote the circle centered at B which passes through C.
Let $\mathcal{O}Cb$ denote the circle centered at C which passes through B.
Let $\mathcal{O}Ab$ intersect line segment AC at point D.
Let $\mathcal{O}Ac$ intersect $\mathcal{O}Cb$ at a point E.
Let $\mathcal{O}Ac$ intersect $\mathcal{O}Bc$ at a point F.

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3. Results

Theorem 3.1. *B, D, and E are collinear if and only if B, D, and F are collinear.*

Proof. $\mathcal{O}Ab$ is given by the equation $x^2 + y^2 = 1$.
$\mathcal{O}Ba$ is given by the equation $(x-1)^2 + y^2 = 1$.
$\mathcal{O}Ac$ is given by the equation $x^2 + y^2 = x_C^2 + y_C^2$.
$\mathcal{O}Ca$ is given by the equation $(x - x_C)^2 + (y - y_C)^2 = x_C^2 + y_C^2$.
$\mathcal{O}Bc$ is given by the equation $(x-1)^2 + y^2 = (x_C - 1)^2 + y_C^2$.
$\mathcal{O}Cb$ is given by the equation $(x - x_C)^2 + (y - y_C)^2 = (x_C - 1)^2 + y_C^2$.
Line AC is given by the equation $y = x(\frac{y_C}{x_C})$.
D has coordinates given by $x_D = \frac{x_C}{\sqrt{x_C^2+y_C^2}}$ and $y_D = \frac{y_C}{\sqrt{x_C^2+y_C^2}}$.
The coordinates of F are given by $x_F = x_C$ and $y_F = -y_C$.
Line DB is given by the equation $y = \frac{y_C(x-1)}{x_C - \sqrt{x_C^2+y_C^2}}$.
Suppose that D, B, and F were collinear. Then this would force that
$y_F - \frac{y_C(x_F-1)}{x_C-\sqrt{x_C^2+y_C^2}} = 0 \iff -y_C - \frac{y_C(x_C-1)}{x_C-\sqrt{x_C^2+y_C^2}} = 0 \implies y_C = \sqrt{(x_C-1)(3x_C-1)}$.
The coordinates of E are then given by
$x_E = \frac{x_C^2-(x_C-1)(3x_C-1)}{2x_C-1}$ and $y_E = \frac{2x_C\sqrt{(x_C-1)(3x_C-1)}}{2x_C-1}$.
Then it follows that DB is given by the equation
$y = -\frac{(x-1)\sqrt{(x_C-1)(3x_C-1)}}{x_C-1}$ and we calculate that
$y_E + \frac{(x_E-1)\sqrt{(x_C-1)(3x_C-1)}}{x_C-1} = \frac{2x_C\sqrt{(x_C-1)(3x_C-1)}}{2x_C-1} + \frac{(\frac{x_C^2-(x_C-1)(3x_C-1)}{2x_C-1}-1)\sqrt{(x_C-1)(3x_C-1)}}{x_C-1} =$
0 which indicates that E indeed lies on line DB. □

References

[1] Warren, B.L., "Intersections of Circles Centered at Triangle Vertices Passing Through the Other Vertices", *International Journal of Computer Discovered Mathematics*, Vol 8, paper 8, pp. 38-41.

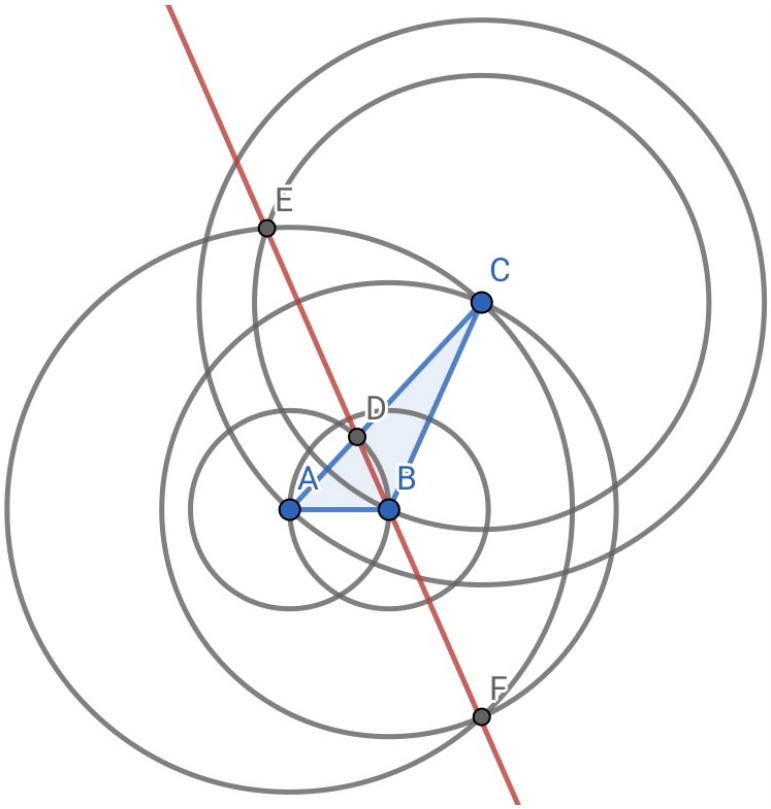

Figure 1. E lies on line BD if and only if F lies on line BD.