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Equilateral Triangles Formed from Equilateral Triangles Erected on Triangles

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1. Introduction

Napoleon's theorem [1] states that if equilateral triangles are erected on the sides of a triangle, then their centers form an equilateral triangle. Here, we will demonstrate the existence of several other equilateral triangles that can be found from this construction.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let A, C, and D form an equilateral triangle (internal or external).
Let B, C, and E form an equilateral triangle (internal or external).
Let A, B, and F form an equilateral triangle (internal or external).
Let G be the midpoint of D and E.
Let H be the midpoint of D and F.
Let I be the midpoint of E and F.
Let K be the circumcenter of A, G, and I.
Let L be the circumcenter of C, H, and I.
Let M be the circumcenter of B, G, and H.
Then,
1. triangles AGI, BGH, and CHI are equilateral,
2. lines AH, BI, and CG are concurrent at point J and intersect at 60 degree angles,

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3. circles AGI, BGH, and CHI are concurrent at point J, and
4. K, L, and M form an equilateral triangle.

Proof. Without loss of generality, we will use Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
We will split the proof into two cases: where the triangles are external and internal.
Case 1: the triangles are external.
D has coordinates
$(x_C/2 - (3^{1/2} * y_C)/2, y_C/2 + (3^{1/2} * x_C)/2)$.
E has coordinates
$(x_C/2 + (3^{1/2} * y_C)/2 + 1/2, (y_C - 3^{1/2} * (x_C - 1))/2)$.
F has coordinates $(1/2, -3^{1/2}/2)$.
G has coordinates $((x_C + 1/2)/2, (2 * y_C + 3^{1/2})/4)$.
H has coordinates $((-3^{1/2} * y_C + x_C + 1)/4, (y_C + 3^{1/2} * (x_C - 1))/4)$.
I has coordinates $((3^{1/2} * y_C + x_C + 2)/4, (y_C - 3^{1/2} * x_C)/4)$.
K has coordinates
$((y_C * (y_C * (y_C + 3^{1/2} * (x_C + 2)) + (x_C + 2)^2) + 3^{1/2} * (x_C + 1) * (x_C^2 + x_C + 1))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1))), (y_C * (y_C * (3^{1/2} * y_C - x_C + 4) + 3^{1/2} * (x_C^2 + 2)) - x_C^3 + 1)/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1))))$.
L has coordinates
$((2 * x_C + 1)/4, (y_C * (y_C * (2 * 3^{3/2} * y_C + 3) + 2 * 3^{3/2} * (x_C - 1) * x_C + 3^{1/2}) - 3 * x_C^2 + 3 * x_C - 1)/(4 * (y_C * (3^{3/2} * y_C + 3) + 3^{3/2} * (x_C - 1) * x_C + 3^{1/2})))$.
M has coordinates
$(-(y_C * (y_C * (y_C - 3^{1/2} * (x_C + 1)) + x_C^2 - 6 * x_C - 3) - 3^{1/2} * (x_C + 2) * (x_C^2 - 3 * x_C + 3))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 - 3 * x_C + 3))), (y_C * (y_C * (3^{1/2} * y_C + x_C + 3) + 3^{1/2} * (x_C - 2) * x_C + 3^{3/2}) + x_C * (x_C^2 - 3 * x_C + 3))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 - 3 * x_C + 3))))$.
We verify that A, G, and I form an equilateral triangle since we calculate that
$((x_C + 1/2)/2)^2 + ((2 * y_C + 3^{1/2})/4)^2 - (((3^{1/2} * y_C + x_C + 2)/4)^2 + ((y_C - 3^{1/2} * x_C)/4)^2) = 0$
and
$((x_C + 1/2)/2)^2 + ((2 * y_C + 3^{1/2})/4)^2 - (((x_C + 1/2)/2 - (3^{1/2} * y_C + x_C + 2)/4)^2 + ((2 * y_C + 3^{1/2})/4 - (y_C - 3^{1/2} * x_C)/4)^2) = 0$.
We verify that B, G, and H form an equilateral triangle since we calculate that
$((x_C + 1/2)/2 - 1)^2 + ((2 * y_C + 3^{1/2})/4)^2 - (((-3^{1/2} * y_C + x_C + 1)/4 - 1)^2 + ((y_C + 3^{1/2} * (x_C - 1))/4)^2) = 0$
and
$((x_C + 1/2)/2 - 1)^2 + ((2 * y_C + 3^{1/2})/4)^2 - (((x_C + 1/2)/2 - (-3^{1/2} * y_C + x_C + 1)/4)^2 + ((2 * y_C + 3^{1/2})/4 - (y_C + 3^{1/2} * (x_C - 1))/4)^2) = 0$.
We verify that C, H, and I form an equilateral triangle since we calculate that
$(x_C - (-3^{1/2} * y_C + x_C + 1)/4)^2 + (y_C - (y_C + 3^{1/2} * (x_C - 1))/4)^2 - ((x_C - (3^{1/2} * y_C + x_C + 2)/4)^2 + (y_C - (y_C - 3^{1/2} * x_C)/4)^2) = 0$
and
$(x_C - (-3^{1/2} * y_C + x_C + 1)/4)^2 + (y_C - (y_C + 3^{1/2} * (x_C - 1))/4)^2 - (((-3^{1/2} * y_C + x_C + 1)/4 - (3^{1/2} * y_C + x_C + 2)/4)^2 + ((y_C + 3^{1/2} * (x_C - 1))/4 - (y_C - 3^{1/2} * x_C)/4)^2) = 0$.
Line AH is given by the equation
$y = ((y_C + 3^{1/2} * (x_C - 1)) * x)/(-3^{1/2} * y_C + x_C + 1)$.

Line BI is given by the equation
$y = ((y_C - 3^{1/2} * x_C) * (x - 1))/(3^{1/2} * y_C + x_C - 2)$.
Line CG is given by the equation
$y = (y_C * (2 * x - 1) - 3^{1/2} * (x - x_C))/(2 * x_C - 1)$.
Circle AGI is given by the equation
$(x - ((y_C * (y_C * (y_C + 3^{1/2} * (x_C + 2)) + (x_C + 2)^2) + 3^{1/2} * (x_C + 1) * (x_C^2 + x_C + 1))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1)))))^2 + (y - ((y_C * (y_C * (3^{1/2} * y_C - x_C + 4) + 3^{1/2} * (x_C^2 + 2)) - x_C^3 + 1)/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1)))))^2 - (((y_C * (y_C * (y_C + 3^{1/2} * (x_C + 2)) + (x_C + 2)^2) + 3^{1/2} * (x_C + 1) * (x_C^2 + x_C + 1))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1))))^2 + ((y_C * (y_C * (3^{1/2} * y_C - x_C + 4) + 3^{1/2} * (x_C^2 + 2)) - x_C^3 + 1)/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1))))^2) = 0$.
Circle BGH is given by the equation
$(x - (-(y_C * (y_C * (y_C - 3^{1/2} * (x_C + 1)) + x_C^2 - 6 * x_C - 3) - 3^{1/2} * (x_C + 2) * (x_C^2 - 3 * x_C + 3))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 - 3 * x_C + 3)))))^2 + (y - ((y_C * (y_C * (3^{1/2} * y_C + x_C + 3) + 3^{1/2} * (x_C - 2) * x_C + 3^{3/2}) + x_C * (x_C^2 - 3 * x_C + 3))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 - 3 * x_C + 3)))))^2 - (((-(y_C * (y_C * (y_C - 3^{1/2} * (x_C + 1)) + x_C^2 - 6 * x_C - 3) - 3^{1/2} * (x_C + 2) * (x_C^2 - 3 * x_C + 3))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 - 3 * x_C + 3)))) - 1)^2 + ((y_C * (y_C * (3^{1/2} * y_C + x_C + 3) + 3^{1/2} * (x_C - 2) * x_C + 3^{3/2}) + x_C * (x_C^2 - 3 * x_C + 3))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 - 3 * x_C + 3))))^2) = 0$.
Circle CHI is given by the equation
$(x - ((2 * x_C + 1)/4))^2 + (y - ((y_C * (y_C * (2 * 3^{3/2} * y_C + 3) + 2 * 3^{3/2} * (x_C - 1) * x_C + 3^{1/2}) - 3 * x_C^2 + 3 * x_C - 1)/(4 * (y_C * (3^{3/2} * y_C + 3) + 3^{3/2} * (x_C - 1) * x_C + 3^{1/2})))))^2 - ((y_C - ((y_C * (y_C * (2 * 3^{3/2} * y_C + 3) + 2 * 3^{3/2} * (x_C - 1) * x_C + 3^{1/2}) - 3 * x_C^2 + 3 * x_C - 1)/(4 * (y_C * (3^{3/2} * y_C + 3) + 3^{3/2} * (x_C - 1) * x_C + 3^{1/2}))))^2 + (x_C - ((2 * x_C + 1)/4))^2) = 0$.
Consider the point J whose coordinates are given by
$(((y_C - 3^{1/2} * x_C) * (3^{1/2} * y_C - x_C - 1))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1))), -((y_C + 3^{1/2} * (x_C - 1)) * (y_C - 3^{1/2} * x_C))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1))))$.
We verify that point J lies on the above lines and circles since we calculate that
$((y_C + 3^{1/2} * (x_C - 1)) * (((y_C - 3^{1/2} * x_C) * (3^{1/2} * y_C - x_C - 1))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))))/(-3^{1/2} * y_C + x_C + 1) - (-((y_C + 3^{1/2} * (x_C - 1)) * (y_C - 3^{1/2} * x_C))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) = 0$,
$((y_C - 3^{1/2} * x_C) * ((((y_C - 3^{1/2} * x_C) * (3^{1/2} * y_C - x_C - 1))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) - 1))/(3^{1/2} * y_C + x_C - 2) - (-((y_C + 3^{1/2} * (x_C - 1)) * (y_C - 3^{1/2} * x_C))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) = 0$,
$(y_C * (2 * (((y_C - 3^{1/2} * x_C) * (3^{1/2} * y_C - x_C - 1))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) - 1) - 3^{1/2} * ((((y_C - 3^{1/2} * x_C) * (3^{1/2} * y_C - x_C - 1))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) - x_C))/(2 * x_C - 1) - (-((y_C + 3^{1/2} * (x_C - 1)) * (y_C - 3^{1/2} * x_C))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) = 0$,
$((((y_C - 3^{1/2} * x_C) * (3^{1/2} * y_C - x_C - 1))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) - ((y_C * (y_C * (y_C + 3^{1/2} * (x_C + 2)) + (x_C + 2)^2) + 3^{1/2} * (x_C + 1) * (x_C^2 + x_C + 1))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1)))))^2 + ((-((y_C + 3^{1/2} * (x_C - 1)) * (y_C - 3^{1/2} * x_C))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C + 1)))) - ((y_C * (y_C * (3^{1/2} * y_C - x_C + 4) + 3^{1/2} * (x_C^2 + 2)) - x_C^3 + 1)/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1)))))^2 - (((y_C * (y_C * (y_C + 3^{1/2} * (x_C + 2)) + (x_C + 2)^2) + 3^{1/2} * (x_C + 1) * (x_C^2 + x_C + 1))/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1))))^2 + ((y_C * (y_C * (3^{1/2} * y_C - x_C + 4) + 3^{1/2} * (x_C^2 + 2)) - x_C^3 + 1)/(4 * (y_C * (3^{1/2} * y_C + 3) + 3^{1/2} * (x_C^2 + x_C + 1))))^2) = 0$,
$((((y_C - 3^{1/2} * x_C) * (3^{1/2} * y_C - x_C - 1))/(2 * (y_C * (3^{1/2} * y_C - 3) + 3^{1/2} * (x_C^2 - x_C +$

$1))))-(-(y_C*(y_C*(y_C-3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)-3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3)))))^2+((-((y_C+3^{1/2}*(x_C-1))*(y_C-3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*(y_C*(3^{1/2}*y_C+x_C+3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})+x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3)))))^2-(((-(y_C*(y_C*(y_C-3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)-3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3))))-1)^2+((y_C*(y_C*(3^{1/2}*y_C+x_C+3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})+x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3))))^2)=0,$
and
$((((y_C-3^{1/2}*x_C)*(3^{1/2}*y_C-x_C-1))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-((2*x_C+1)/4))^2+((-((y_C+3^{1/2}*(x_C-1))*(y_C-3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*(y_C*(2*3^{3/2}*y_C+3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})-3*x_C^2+3*x_C-1)/(4*(y_C*(3^{3/2}*y_C+3)+3^{3/2}*(x_C-1)*x_C+3^{1/2}))))^2-((y_C-((y_C*(y_C*(2*3^{3/2}*y_C+3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})-3*x_C^2+3*x_C-1)/(4*(y_C*(3^{3/2}*y_C+3)+3^{3/2}*(x_C-1)*x_C+3^{1/2}))))^2+(x_C-((2*x_C+1)/4))^2)=0.$
To show that lines AH, BI, and CG are concurrent at 60 degree angles, we will take a line parallel to AH, intersect it with lines BI and CG to form two points N and O, and then will show that J, N, and O form an equilateral triangle.
Consider the line whose equation is given by
$y=((y_C+3^{1/2}*(x_C-1))*x)/(-3^{1/2}*y_C+x_C+1)+1.$
We get that N has coordinates
$(((3^{1/2}*y_C-x_C-1)*(y_C*((3^{1/2}+1)*y_C+4*x_C-3^{1/2}-5)+x_C*((3^{1/2}-3)*x_C-3^{3/2}+3)+2*3^{1/2}))/(2*(y_C+3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))),((y_C-3^{1/2}*x_C)*((3^{1/2}-1)*y_C-(3^{1/2}+1)*x_C+3^{1/2}-1))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))$
and O has coordinates
$(((3^{1/2}*y_C-x_C-1)*(y_C*(y_C+2*x_C-3^{1/2}-1)+x_C*((2*3^{1/2}-3)*x_C-3^{3/2}+3)+3^{1/2}))/(2*(y_C+3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))),(y_C*((2*3^{1/2}-1)*y_C-2*x_C+3^{1/2}-5)+x_C*(3*x_C+3^{1/2}-3)+3^{1/2})/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))).$
We verify that J, N, and O form an equilateral triangle since we calculate that
$((((y_C-3^{1/2}*x_C)*(3^{1/2}*y_C-x_C-1))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-(((3^{1/2}*y_C-x_C-1)*(y_C*((3^{1/2}+1)*y_C+4*x_C-3^{1/2}-5)+x_C*((3^{1/2}-3)*x_C-3^{3/2}+3)+2*3^{1/2}))/(2*(y_C+3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((-((y_C+3^{1/2}*(x_C-1))*(y_C-3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-(((y_C-3^{1/2}*x_C)*((3^{1/2}-1)*y_C-(3^{1/2}+1)*x_C+3^{1/2}-1))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2-(((((y_C-3^{1/2}*x_C)*(3^{1/2}*y_C-x_C-1))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-(((3^{1/2}*y_C-x_C-1)*(y_C*(y_C+2*x_C-3^{1/2}-1)+x_C*((2*3^{1/2}-3)*x_C-3^{3/2}+3)+3^{1/2}))/(2*(y_C+3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((-((y_C+3^{1/2}*(x_C-1))*(y_C-3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*((2*3^{1/2}-1)*y_C-2*x_C+3^{1/2}-5)+x_C*(3*x_C+3^{1/2}-3)+3^{1/2})/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2)=0$
and
$((((y_C-3^{1/2}*x_C)*(3^{1/2}*y_C-x_C-1))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-(((3^{1/2}*y_C-x_C-1)*(y_C*((3^{1/2}+1)*y_C+4*x_C-3^{1/2}-5)+x_C*((3^{1/2}-3)*x_C-3^{3/2}+3)+2*3^{1/2}))/(2*(y_C+3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((-((y_C+3^{1/2}*(x_C-1))*(y_C-3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-(((y_C-3^{1/2}*x_C)*((3^{1/2}-1)*y_C-(3^{1/2}+1)*x_C+3^{1/2}-1))/(2*(y_C*(3^{1/2}*$

$y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2-(((((3^{1/2}*y_C-x_C-1)*(y_C*((3^{1/2}+1)*y_C+4*x_C-3^{1/2}-5)+x_C*((3^{1/2}-3)*x_C-3^{3/2}+3)+2*3^{1/2}))/(2*(y_C+3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-(((3^{1/2}*y_C-x_C-1)*(y_C*(y_C+2*x_C-3^{1/2}-1)+x_C*((2*3^{1/2}-3)*x_C-3^{3/2}+3)+3^{1/2}))/(2*(y_C+3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((((y_C-3^{1/2}*x_C)*((3^{1/2}-1)*y_C-(3^{1/2}+1)*x_C+3^{1/2}-1))/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*((2*3^{1/2}-1)*y_C-2*x_C+3^{1/2}-5)+x_C*(3*x_C+3^{1/2}-3)+3^{1/2})/(2*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-x_C+1)))))^2)=0.$

We verify that K, L, and M form an equilateral triangle since we calculate that

$((y_C*(y_C*(y_C+3^{1/2}*(x_C+2))+(x_C+2)^2)+3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2+x_C+1)))+(y_C*(y_C*(y_C-3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)-3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3))))^2+((y_C*(y_C*(3^{1/2}*y_C-x_C+4)+3^{1/2}*(x_C^2+2))-x_C^3+1)/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(3^{1/2}*y_C+x_C+3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})+x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3))))^2+(((y_C*(y_C*(y_C+3^{1/2}*(x_C+2))+(x_C+2)^2)+3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2+x_C+1)))-(2*x_C+1)/4)^2+((y_C*(y_C*(3^{1/2}*y_C-x_C+4)+3^{1/2}*(x_C^2+2))-x_C^3+1)/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(2*3^{3/2}*y_C+3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})-3*x_C^2+3*x_C-1)/(4*(y_C*(3^{3/2}*y_C+3)+3^{3/2}*(x_C-1)*x_C+3^{1/2})))^2)=0$

and

$((y_C*(y_C*(y_C+3^{1/2}*(x_C+2))+(x_C+2)^2)+3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2+x_C+1)))+(y_C*(y_C*(y_C-3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)-3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3))))^2+((y_C*(y_C*(3^{1/2}*y_C-x_C+4)+3^{1/2}*(x_C^2+2))-x_C^3+1)/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(3^{1/2}*y_C+x_C+3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})+x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3))))^2-((-(y_C*(y_C*(y_C-3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)-3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3)))-(2*x_C+1)/4)^2+((y_C*(y_C*(3^{1/2}*y_C+x_C+3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})+x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-3*x_C+3)))-(y_C*(y_C*(2*3^{3/2}*y_C+3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})-3*x_C^2+3*x_C-1)/(4*(y_C*(3^{3/2}*y_C+3)+3^{3/2}*(x_C-1)*x_C+3^{1/2})))^2)=0$ as desired.

Case 2: the triangles are internal.

D has coordinates

$(x_C/2+(3^{1/2}*y_C)/2, y_C/2-(3^{1/2}*x_C)/2).$

E has coordinates

$(x_C/2-(3^{1/2}*y_C)/2+1/2, (y_C+3^{1/2}*(x_C-1))/2)$

F has coordinates $(1/2, 3^{1/2}/2).$

G has coordinates $((x_C+1/2)/2, (2*y_C-3^{1/2})/4).$

H has coordinates $((3^{1/2}*y_C+x_C+1)/4, (y_C-3^{1/2}*(x_C-1))/4).$

I has coordinates $((-3^{1/2}*y_C+x_C+2)/4, (y_C+3^{1/2}*x_C)/4).$

K has coordinates

$(-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1))), (y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))).$

L has coordinates

$((2*x_C+1)/4, (y_C*(y_C*(2*3^{3/2}*y_C-3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})+3*x_C^2-3*x_C+1)/(4*(y_C*(3^{3/2}*y_C-3)+3^{3/2}*(x_C-1)*x_C+3^{1/2}))).$

M has coordinates

$((y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))), (y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))$.
We verify that A, G, and I form an equilateral triangle since we calculate that
$((x_C+1/2)/2)^2+((2*y_C-3^{1/2})/4)^2-(((-3^{1/2}*y_C+x_C+2)/4)^2+((y_C+3^{1/2}*x_C)/4)^2)=0$
and
$((x_C+1/2)/2)^2+((2*y_C-3^{1/2})/4)^2-(((x_C+1/2)/2-(-3^{1/2}*y_C+x_C+2)/4)^2+((2*y_C-3^{1/2})/4-(y_C+3^{1/2}*x_C)/4)^2)=0$.
We verify that B, G, and H form an equilateral triangle since we calculate that
$((x_C+1/2)/2-1)^2+((2*y_C-3^{1/2})/4)^2-(((3^{1/2}*y_C+x_C+1)/4-1)^2+((y_C-3^{1/2}*(x_C-1))/4)^2)=0$
and
$((x_C+1/2)/2-1)^2+((2*y_C-3^{1/2})/4)^2-(((x_C+1/2)/2-(3^{1/2}*y_C+x_C+1)/4)^2+((2*y_C-3^{1/2})/4-(y_C-3^{1/2}*(x_C-1))/4)^2)=0$.
We verify that C, H, and I form an equilateral triangle since we calculate that
$(x_C-(3^{1/2}*y_C+x_C+1)/4)^2+(y_C-((y_C-3^{1/2}*(x_C-1))/4))^2-((x_C-(-3^{1/2}*y_C+x_C+2)/4)^2+(y_C-(y_C+3^{1/2}*x_C)/4)^2)=0$
and
$(x_C-(3^{1/2}*y_C+x_C+1)/4)^2+(y_C-((y_C-3^{1/2}*(x_C-1))/4))^2-(((3^{1/2}*y_C+x_C+1)/4-(-3^{1/2}*y_C+x_C+2)/4)^2+(((y_C-3^{1/2}*(x_C-1))/4)-(y_C+3^{1/2}*x_C)/4)^2)=0$.
Line AH is given by the equation
$y=((y_C-3^{1/2}*(x_C-1))*x)/(3^{1/2}*y_C+x_C+1)$.
Line BI is given by the equation
$y=((y_C+3^{1/2}*x_C)*(x-1))/(-3^{1/2}*y_C+x_C-2)$.
Line CG is given by the equation
$y=(y_C*(2*x-1)+3^{1/2}*(x-x_C))/(2*x_C-1)$.
Circle AGI is given by the equation
$(x-(-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))))^2+(y-((y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))))^2-((-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1))))^2+((y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1))))^2)=0$.
Circle BGH is given by the equation
$(x-(((y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))))^2+(y-((y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3)))))^2-(((((y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))-1)^2+((y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))^2)=0$.
Circle CHI is given by the equation
$(x-((2*x_C+1)/4))^2+(y-((y_C*(y_C*(2*3^{3/2}*y_C-3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})+3*x_C^2-3*x_C+1)/(4*(y_C*(3^{3/2}*y_C-3)+3^{3/2}*(x_C-1)*x_C+3^{1/2})))))^2-((y_C-((y_C*(y_C*(2*3^{3/2}*y_C-3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})+3*x_C^2-3*x_C+1)/(4*(y_C*(3^{3/2}*y_C-3)+3^{3/2}*(x_C-1)*x_C+3^{1/2}))))^2+(x_C-((2*x_C+1)/4))^2)=0$.
Consider the point J whose coordinates are given by
$(((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-$

$x_C+1))), ((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))$.
We verify that point J lies on the above lines and circles since we calculate that
$((y_C-3^{1/2}*(x_C-1))*(((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))/(3^{1/2}*y_C+x_C+1)-(((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))=0,$
$((y_C+3^{1/2}*x_C)*((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-1))/(-3^{1/2}*y_C+x_C-2)-(((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))=0,$
$(y_C*(2*(((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-1)+3^{1/2}*((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-x_C))/(2*x_C-1)-(((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))=0,$
$((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1))))^2+((((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1))))^2-((-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1))))^2+((y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1))))^2)=0,$
$((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(((y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))))^2+((((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3)))))^2-(((((y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))-1)^2+((y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))^2)=0,$
and
$((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-((2*x_C+1)/4))^2+((((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*(y_C*(2*3^{3/2}*y_C-3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})+3*x_C^2-3*x_C+1)/(4*(y_C*(3^{3/2}*y_C-3)+3^{3/2}*(x_C-1)*x_C+3^{1/2}))))^2-((y_C-((y_C*(y_C*(2*3^{3/2}*y_C-3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})+3*x_C^2-3*x_C+1)/(4*(y_C*(3^{3/2}*y_C-3)+3^{3/2}*(x_C-1)*x_C+3^{1/2}))))^2+(x_C-((2*x_C+1)/4))^2)=0.$
To show that lines AH, BI, and CG are concurrent at 60 degree angles, we will take a line parallel to AH, intersect it with lines BI and CG to form two points N and O, and then will show that J, N, and O form an equilateral triangle.
Consider the line whose equation is given by
$y=((y_C-3^{1/2}*(x_C-1))*x)/(3^{1/2}*y_C+x_C+1)+1.$
We get that N has coordinates
$(-((3^{1/2}*y_C+x_C+1)*(y_C*((3^{1/2}-1)*y_C-4*x_C-3^{1/2}+5)+x_C*((3^{1/2}+3)*x_C-3^{3/2}-3)+2*3^{1/2}))/(2*(y_C-3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))), ((y_C+3^{1/2}*x_C)*((3^{1/2}+1)*y_C-(3^{1/2}-1)*x_C+3^{1/2}+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))$
and O has coordinates

$(((3^{1/2}*y_C+x_C+1)*(y_C*(y_C+2*x_C+3^{1/2}-1)-x_C*((2*3^{1/2}+3)*x_C-3^{3/2}-3)-3^{1/2}))/(2*(y_C-3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))),(y_C*((2*3^{1/2}+1)*y_C+2*x_C+3^{1/2}+5)-x_C*(3*x_C-3^{1/2}-3)+3^{1/2})/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))).$

We verify that J, N, and O form an equilateral triangle since we calculate that

$((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(-((3^{1/2}*y_C+x_C+1)*(y_C*((3^{1/2}-1)*y_C-4*x_C-3^{1/2}+5)+x_C*((3^{1/2}+3)*x_C-3^{3/2}-3)+2*3^{1/2}))/(2*(y_C-3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(((y_C+3^{1/2}*x_C)*((3^{1/2}+1)*y_C-(3^{1/2}-1)*x_C+3^{1/2}+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2-(((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(((3^{1/2}*y_C+x_C+1)*(y_C*(y_C+2*x_C+3^{1/2}-1)-x_C*((2*3^{1/2}+3)*x_C-3^{3/2}-3)-3^{1/2}))/(2*(y_C-3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(((y_C*((2*3^{1/2}+1)*y_C+2*x_C+3^{1/2}+5)-x_C*(3*x_C-3^{1/2}-3)+3^{1/2})/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2)=0$

and

$((((y_C+3^{1/2}*x_C)*(3^{1/2}*y_C+x_C+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(-((3^{1/2}*y_C+x_C+1)*(y_C*((3^{1/2}-1)*y_C-4*x_C-3^{1/2}+5)+x_C*((3^{1/2}+3)*x_C-3^{3/2}-3)+2*3^{1/2}))/(2*(y_C-3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((((y_C-3^{1/2}*(x_C-1))*(y_C+3^{1/2}*x_C))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(((y_C+3^{1/2}*x_C)*((3^{1/2}+1)*y_C-(3^{1/2}-1)*x_C+3^{1/2}+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2-(((-((3^{1/2}*y_C+x_C+1)*(y_C*((3^{1/2}-1)*y_C-4*x_C-3^{1/2}+5)+x_C*((3^{1/2}+3)*x_C-3^{3/2}-3)+2*3^{1/2}))/(2*(y_C-3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-(((3^{1/2}*y_C+x_C+1)*(y_C*(y_C+2*x_C+3^{1/2}-1)-x_C*((2*3^{1/2}+3)*x_C-3^{3/2}-3)-3^{1/2}))/(2*(y_C-3^{1/2}*(x_C-1))*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2+((((y_C+3^{1/2}*x_C)*((3^{1/2}+1)*y_C-(3^{1/2}-1)*x_C+3^{1/2}+1))/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1))))-((y_C*((2*3^{1/2}+1)*y_C+2*x_C+3^{1/2}+5)-x_C*(3*x_C-3^{1/2}-3)+3^{1/2})/(2*(y_C*(3^{1/2}*y_C+3)+3^{1/2}*(x_C^2-x_C+1)))))^2)=0.$

We verify that K, L, and M form an equilateral triangle since we calculate that

$(-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))^2+((y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))^2-((-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))-(2*x_C+1)/4)^2+((y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(2*3^{3/2}*y_C-3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})+3*x_C^2-3*x_C+1)/(4*(y_C*(3^{3/2}*y_C-3)+3^{3/2}*(x_C-1)*x_C+3^{1/2})))^2)=0$

and

$(-(y_C*(y_C*(y_C-3^{1/2}*(x_C+2))+(x_C+2)^2)-3^{1/2}*(x_C+1)*(x_C^2+x_C+1))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))^2+((y_C*(y_C*(3^{1/2}*y_C+x_C-4)+3^{1/2}*(x_C^2+2))+x_C^3-1)/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2+x_C+1)))-(y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3))))^2-(((y_C*(y_C*(y_C+3^{1/2}*(x_C+1))+x_C^2-6*x_C-3)+3^{1/2}*(x_C+2)*(x_C^2-3*x_C+3))/(4*$

$(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3)))-(2*x_C+1)/4)^2+((y_C*(y_C*(3^{1/2}*y_C-x_C-3)+3^{1/2}*(x_C-2)*x_C+3^{3/2})-x_C*(x_C^2-3*x_C+3))/(4*(y_C*(3^{1/2}*y_C-3)+3^{1/2}*(x_C^2-3*x_C+3)))-(y_C*(y_C*(2*3^{3/2}*y_C-3)+2*3^{3/2}*(x_C-1)*x_C+3^{1/2})+3*x_C^2-3*x_C+1)/(4*(y_C*(3^{3/2}*y_C-3)+3^{3/2}*(x_C-1)*x_C+3^{1/2})))^2)=0$ as desired. □

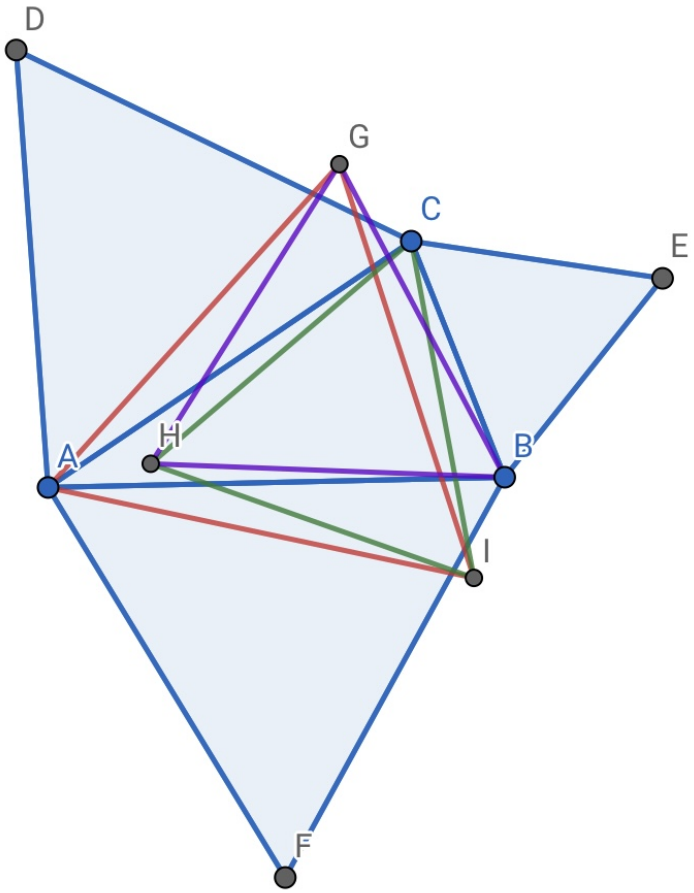

Figure 1. Triangles AGI, BGH, and CHI are equilateral.

References

[1] Weisstein, Eric W. "Napoleon's Theorem." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Orthocenter.html

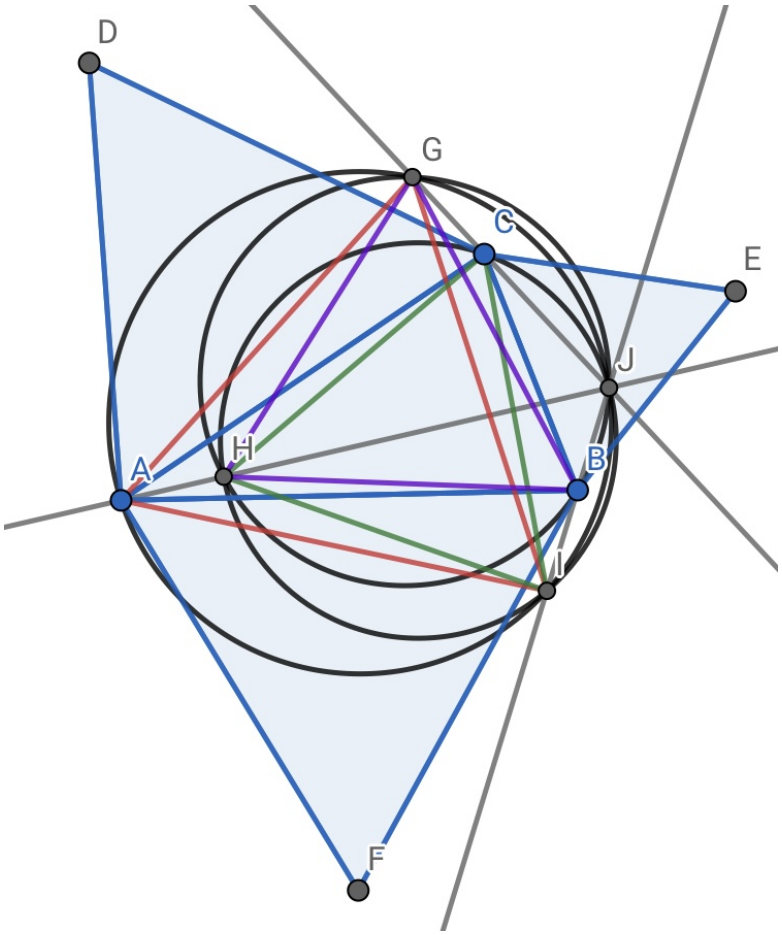

Figure 2. Lines AH, BI, and CG are concurrent at point J and intersect at 60 degree angles, and circles AGI, BGH, and CHI are concurrent at point J.

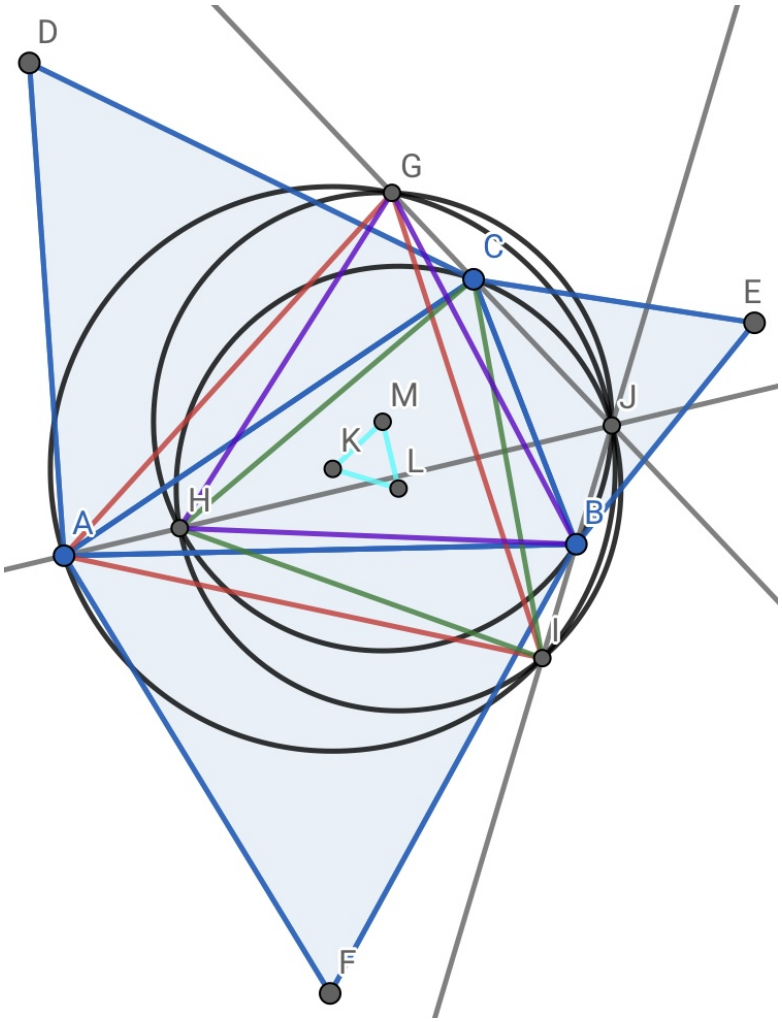

Figure 3. The circumcenters of AGI, BGH, and CHI form an equilateral triangle.