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The Perspector of the Circumcevian and Antiorthocevian Triangles

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1. Introduction

Consider three concurrent cevians. If each cevian is extended to intersect the circumcircle, the triangle formed from the three resulting points is known as the circumcevian triangle [1]. Now, if we consider the anticevian triangle where each line is orthogonal to the respective cevian, we will call the resulting triangle between these points the antiorthocevian triangle. Here, we will prove that these two triangles are in perspective.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be some point in the plane.
Let line CD intersect the circumcircle of ΔABC at point E.
Let line BD intersect the circumcircle of ΔABC at point F.
Let line AD intersect the circumcircle of ΔABC at point G.
Let α be the line passing through A which is orthogonal to line AD.
Let β be the line passing through B which is orthogonal to line BD.
Let γ be the line passing through C which is orthonal to line CD.
Let α and β intersect at point H.
Let α and γ intersect at point I.
Let β and γ intersect at point J.
Then lines EH, FI, and GJ are concurrent.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
The circumcircle of A, B, and C is given by the equation
$(x - \frac{1}{2})^2 + (y - \frac{x_C(x_C-1)+y_C^2}{2y_C})^2 = \frac{1}{4} + (\frac{x_C(x_C-1)+y_C^2}{2y_C})^2$.
E has coordinates
$(((x_Cy_D - x_Dy_C)(y_C(y_D - y_C) + (x_C - 1)(x_D - x_C)))/(y_C((y_D - y_C)^2 + (x_D - x_C)^2)), (((x_C-1)y_D-(x_D-1)y_C)(x_Cy_D-x_Dy_C))/(y_C((y_D-y_C)^2+(x_D-x_C)^2)))$.
F has coordinates
$((y_D(y_C(y_D+(x_D-1)y_C)+(x_C-1)x_C(x_D-1)))/(y_C(y_D^2+(x_D-1)^2)), (y_D((y_C^2+(x_C-1)x_C)y_D-(x_D-1)y_C))/(y_C(y_D^2+(x_D-1)^2)))$.
G has coordinates
$((x_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C))/(y_C(y_D^2+x_D^2)), (y_D((y_C^2+(x_C-1)x_C)y_D+x_Dy_C))/(y_C(y_D^2+x_D^2)))$.
α is given by the equation $y = x(-\frac{x_D}{y_D})$.
β is given by the equation $y = (x - 1)(\frac{1-x_D}{y_D})$.
γ is given by the equation $y = y_C + (x - x_C)(\frac{x_D-x_C}{y_C-y_D})$.
H has coordinates $(1 - x_D, (x_D * (x_D - 1))/y_D)$.
I has coordinates
$(-(y_D * (y_C * (y_D - y_C) + x_C * (x_D - x_C)))/(x_C * y_D - x_D * y_C), (x_D * (y_C * (y_D - y_C) + x_C * (x_D - x_C)))/(x_C * y_D - x_D * y_C))$.
J has coordinates
$(-(y_D*(y_C*(y_D-y_C)+(x_C-1)*x_D-x_C^2+1)+(x_D-1)*y_C)/((x_C-1)*y_D-(x_D-1)*y_C), ((x_D-1)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/((x_C-1)*y_D-(x_D-1)*y_C))$.
Line EH is given by the equation
$y = (((((x_C-1)*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C))/(y_C*((y_D-y_C)^2+(x_D-x_C)^2))-((x_D-1)*x_D)/y_D)*(x+x_D-1))/(((x_C*y_D-x_D*y_C)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/(y_C*((y_D-y_C)^2+(x_D-x_C)^2))+x_D-1)+((x_D-1)*x_D)/y_D$.
Line FI is given by the equation
$y = (((y_D*((y_C^2+(x_C-1)*x_C)*y_D-(x_D-1)*y_C))/(y_C*(y_D^2+(x_D-1)^2))-(x_D*(y_C*(y_D-y_C)+x_C*(x_D-x_C)))/(x_C*y_D-x_D*y_C))*(x+(y_D*(y_C*(y_D-y_C)+x_C*(x_D-x_C)))/(x_C*y_D-x_D*y_C)))/((y_D*(y_C*(y_D+(x_D-1)*y_C)+(x_C-1)*x_C*(x_D-1)))/(y_C*(y_D^2+(x_D-1)^2))+(y_D*(y_C*(y_D-y_C)+x_C*(x_D-x_C)))/(x_C*y_D-x_D*y_C))+(x_D*(y_C*(y_D-y_C)+x_C*(x_D-x_C)))/(x_C*y_D-x_D*y_C)$.
Line GJ is given by the equation
$y = (((y_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))-((x_D-1)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/((x_C-1)*y_D-(x_D-1)*y_C))*(x+(y_D*(y_C*(y_D-y_C)+(x_C-1)*x_D-x_C^2+1)+(x_D-1)*y_C)/((x_C-1)*y_D-(x_D-1)*y_C)))/((x_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))+(y_D*(y_C*(y_D-y_C)+(x_C-1)*x_D-x_C^2+1)+(x_D-1)*y_C)/((x_C-1)*y_D-(x_D-1)*y_C))+((x_D-1)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/((x_C-1)*y_D-(x_D-1)*y_C)$.
Lines EH and FI intersect at the point whose coordinates are given by
$((y_C*((x_D+x_C-1)*y_C^2+(x_C-1)*x_C^2)*y_D^4-(y_C^2*((3*x_D+x_C-2)*y_C^2-x_D*((2*x_C-1)*x_D-x_C^2-2*x_C+2)+(2*x_C-1)*(x_C^2-x_C-1))-(x_C-1)^2*x_C^2*(2*x_D-x_C-1))*y_D^3+y_C*(y_C^2*((2*x_D-1)*y_C^2+x_D*(x_D*(x_D-4*x_C+1)+4*x_C^2+x_C-4)-2*x_C^2-x_C+2)+(x_C-1)*x_C*(x_D^3-4*x_C*x_D^2+x_D^2+2*x_C^2*x_D+3*x_C*x_D-x_C^2-2*x_C))*y_D^2-(2*x_D^3*y_C^4+x_C*x_D^2*y_C^4-3*x_D^2*y_C^4-x_C*x_D*y_C^4-x_D*y_C^4+y_C^4-2*x_C*$

$x_D^4*y_C^2+x_D^4*y_C^2+x_C^2*x_D^3*y_C^2+4*x_C*x_D^3*y_C^2-3*x_D^3*y_C^2+2*x_C^3*x_D^2*y_C^2-4*x_C^2*x_D^2*y_C^2+x_C*x_D^2*y_C^2+x_D^2*y_C^2-2*x_C^3*x_D*y_C^2+x_C^2*x_D*y_C^2-x_C*x_D*y_C^2+x_C^2*y_C^2-x_C^4*x_D^3+2*x_C^3*x_D^3-x_C^2*x_D^3+x_C^5*x_D^2-x_C^4*x_D^2-x_C^3*x_D^2+x_C^2*x_D^2-x_C^5*x_D+2*x_C^4*x_D-x_C^3*x_D)*y_D+(x_D-1)*x_D*(2*x_D-1)*y_C^5-(x_D-1)*x_D*(x_D*((3*x_C-2)*x_D-4*x_C^2+x_C+1)+x_C*(2*x_C-1))*y_C^3+(x_C-1)*x_C*(x_D-1)*x_D*(x_D-x_C)*(x_D^2-2*x_C*x_D+x_C)*y_C)/(((x_C-1)*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C)*(y_D*(y_C*(y_D-y_C)-(x_C-1)*x_C)+(x_D-1)*x_D*y_C)), ((x_C-1)*x_C*(y_C^2+(x_C-1)*x_C)*y_D^5-y_C*(x_D^2*y_C^2+2*x_C*x_D*y_C^2-2*x_D*y_C^2-x_C*y_C^2+2*x_C^3*x_D-3*x_C^2*x_D+x_C*x_D-x_C^3+x_C)*y_D^4+y_C^2*(x_D*(3*(x_D-1)*y_C^2-x_D*((2*x_C-1)*x_D-3*x_C^2+2)-3*x_C^2+2)+x_C)*y_D^3-(x_D-1)*x_D*y_C*(y_C^2*(y_C^2+x_D^2-2*x_C*x_D+2*x_C^2-x_C-3)+(x_C-1)*x_C*(x_D-x_C)^2)*y_D^2+(x_D-1)*x_D*y_C^2*((2*x_D^2-2*x_D-1)*y_C^2-(x_D-x_C)*(x_D*((2*x_C-1)*x_D-2*x_C+2)-x_C))*y_D-(x_D-1)^2*x_D^2*y_C^5+(2*x_C-1)*(x_D-1)^2*x_D^2*(x_D-x_C)*y_C^3-(x_C-1)*x_C*(x_D-1)^2*x_D^2*(x_D-x_C)^2*y_C)/(y_D*((x_C-1)*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C)*(y_D*(y_C*(y_D-y_C)-(x_C-1)*x_C)+(x_D-1)*x_D*y_C)))$.
We then verify that this point lies on line GJ since we calculate
$(((y_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))-((x_D-1)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/((x_C-1)*y_D-(x_D-1)*y_C))*(((y_C*((x_D+x_C-1)*y_C^2+(x_C-1)*x_C^2)*y_D^4-(y_C^2*((3*x_D+x_C-2)*y_C^2-x_D*((2*x_C-1)*x_D-x_C^2-2*x_C+2)+(2*x_C-1)*(x_C^2-x_C-1))-(x_C-1)^2*x_C^2*(2*x_D-x_C-1))*y_D^3+y_C*(y_C^2*((2*x_D-1)*y_C^2+x_D*(x_D*(x_D-4*x_C+1)+4*x_C^2+x_C-4)-2*x_C^2-x_C+2)+(x_C-1)*x_C*(x_D^3-4*x_C*x_D^2+x_D^2+2*x_C^2*x_D+3*x_C*x_D-x_C^2-2*x_C))*y_D^2-(2*x_D^3*y_C^4+x_C*x_D^2*y_C^4-3*x_D^2*y_C^4-x_C*x_D*y_C^4-x_D*y_C^4+y_C^4-2*x_C*x_D^4*y_C^2+x_D^4*y_C^2+x_C^2*x_D^3*y_C^2+4*x_C*x_D^3*y_C^2-3*x_D^3*y_C^2+2*x_C^3*x_D^2*y_C^2-4*x_C^2*x_D^2*y_C^2+x_C*x_D^2*y_C^2+x_D^2*y_C^2-2*x_C^3*x_D*y_C^2+x_C^2*x_D*y_C^2-x_C*x_D*y_C^2+x_C^2*y_C^2-x_C^4*x_D^3+2*x_C^3*x_D^3-x_C^2*x_D^3+x_C^5*x_D^2-x_C^4*x_D^2-x_C^3*x_D^2+x_C^2*x_D^2-x_C^5*x_D+2*x_C^4*x_D-x_C^3*x_D)*y_D+(x_D-1)*x_D*(2*x_D-1)*y_C^5-(x_D-1)*x_D*(x_D*((3*x_C-2)*x_D-4*x_C^2+x_C+1)+x_C*(2*x_C-1))*y_C^3+(x_C-1)*x_C*(x_D-1)*x_D*(x_D-x_C)*(x_D^2-2*x_C*x_D+x_C)*y_C)/(((x_C-1)*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C)*(y_D*(y_C*(y_D-y_C)-(x_C-1)*x_C)+(x_D-1)*x_D*y_C)))+(y_D*(y_C*(y_D-y_C)+(x_C-1)*x_D-x_C^2+1)+(x_D-1)*y_C)/((x_C-1)*y_D-(x_D-1)*y_C)))/((x_D*((y_C^2+(x_C-1)*x_C)*y_D+x_D*y_C))/(y_C*(y_D^2+x_D^2))+(y_D*(y_C*(y_D-y_C)+(x_C-1)*x_D-x_C^2+1)+(x_D-1)*y_C)/((x_C-1)*y_D-(x_D-1)*y_C))+((x_D-1)*(y_C*(y_D-y_C)+(x_C-1)*(x_D-x_C)))/((x_C-1)*y_D-(x_D-1)*y_C)-(((x_C-1)*x_C*(y_C^2+(x_C-1)*x_C)*y_D^5-y_C*(x_D^2*y_C^2+2*x_C*x_D*y_C^2-2*x_D*y_C^2-x_C*y_C^2+2*x_C^3*x_D-3*x_C^2*x_D+x_C*x_D-x_C^3+x_C)*y_D^4+y_C^2*(x_D*(3*(x_D-1)*y_C^2-x_D*((2*x_C-1)*x_D-3*x_C^2+2)-3*x_C^2+2)+x_C)*y_D^3-(x_D-1)*x_D*y_C*(y_C^2*(y_C^2+x_D^2-2*x_C*x_D+2*x_C^2-x_C-3)+(x_C-1)*x_C*(x_D-x_C)^2)*y_D^2+(x_D-1)*x_D*y_C^2*((2*x_D^2-2*x_D-1)*y_C^2-(x_D-x_C)*(x_D*((2*x_C-1)*x_D-2*x_C+2)-x_C))*y_D-(x_D-1)^2*x_D^2*y_C^5+(2*x_C-1)*(x_D-1)^2*x_D^2*(x_D-x_C)*y_C^3-(x_C-1)*x_C*(x_D-1)^2*x_D^2*(x_D-x_C)^2*y_C)/(y_D*((x_C-1)*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C)*(y_D*(y_C*(y_D-y_C)-(x_C-1)*x_C)+(x_D-1)*x_D*y_C)))=0$
as desired. □

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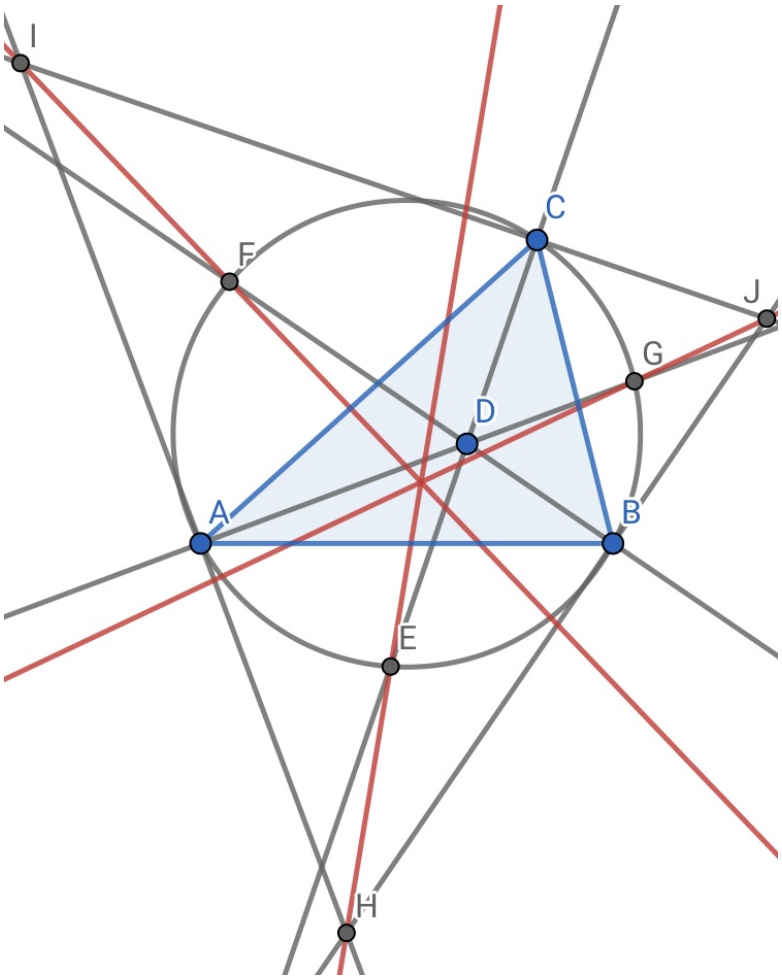

FIGURE 1. The circumcevian and antiorthocevian triangles are in perspective.