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Generalization of the Mittenpunkt

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1. Introduction

The Mittenpunkt [1] is the point of concurrency of the midpoints and excenters of a triangle. It is denoted $X(9)$ in Kimberling's Encyclopedia of Triangle Centers. Here, we will give a generalization of the Mittenpunkt.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be some point in the plane.
Let lines AB and CD intersect at point E.
Let lines AC and BD intersect at point F.
Let lines AD and BC intersect at point G.
Let the circle passing through E, F, and G intersect line AB at point H.
Let the circle passing through E, F, and G intersect line AC at point I.
Let the circle passing through E, F, and G intersect line BC at point J.
Let K be the excenter adjacent to line AB.
Let L be the excenter adjacent to line AC.
Let M be the excenter adjacent to line BC.
Then,
1. Lines KE, LF, and MG are concurrent at point N, and
2. Lines KH, LI, and MJ are concurrent at point O.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $(\frac{x_C y_D - x_D y_C}{y_D - y_C}, 0)$.
F has coordinates $(\frac{x_C y_D}{x_C y_D - x_D y_C + y_C}, \frac{y_C y_D}{x_C y_D - x_D y_C + y_C})$.
G has coordinates $(\frac{x_D y_C}{-x_C y_D + x_D y_C + y_D}, \frac{y_C y_D}{-x_C y_D + x_D y_C + y_D})$.
H has coordinates
$(-(((2x_C-1)y_C^2 - x_C^4 + 2x_C^3 - x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 - (x_C-1)x_C^2(2x_D - 1))y_D^3 + y_C^2((2x_D-1)y_C^2 - 2x_C x_D((3x_C-2)x_D - 2x_C + 1))y_D^2 + 2(2x_C-1)(x_D - 1)x_D^2 y_C^3 y_D - (x_D-1)^2 x_D^2 y_C^4)/(2((x_C-1)y_D - (x_D-1)y_C)((x_C-1)y_D - x_D y_C)(x_C y_D - (x_D-1)y_C)(x_C y_D - x_D y_C)), 0)$.
I has coordinates
$(-(x_C(((2x_C-1)y_C^2 - x_C^4 + 2x_C^3 - x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 - (x_C-1)x_C^2(2x_D - 1))y_D^3 + y_C^2((2x_D-1)y_C^2 - 2x_C x_D((3x_C-2)x_D - 2x_C + 1))y_D^2 + 2(2x_C-1)(x_D - 1)x_D^2 y_C^3 y_D - (x_D-1)^2 x_D^2 y_C^4))/(2(y_C^2 + x_C^2)y_D(y_D - y_C)((x_C-1)y_D - (x_D-1)y_C)((x_C-1)y_D - x_D y_C)), -(y_C(((2x_C-1)y_C^2 - x_C^4 + 2x_C^3 - x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 - (x_C-1)x_C^2(2x_D - 1))y_D^3 + y_C^2((2x_D-1)y_C^2 - 2x_C x_D((3x_C-2)x_D - 2x_C + 1))y_D^2 + 2(2x_C-1)(x_D-1)x_D^2 y_C^3 y_D - (x_D-1)^2 x_D^2 y_C^4))/(2(y_C^2 + x_C^2)y_D(y_D - y_C)((x_C-1)y_D - (x_D-1)y_C)((x_C-1)y_D - x_D y_C)))$.
J has coordinates
$((((4x_C^2 - 3x_C + 1)y_C^2 + x_C^5 - x_C^4 - x_C^3 + x_C^2)y_D^4 - 2y_C(((3x_C-1)x_D + (x_C-1)(2x_C - 1))y_C^2 + 2(x_C-1)^2 x_C^2 x_D)y_D^3 + y_C^2((2x_D(x_D + 3x_C - 2) - 3x_C + 1)y_C^2 + 2(x_C - 1)^2 x_C x_D(3x_D - 2))y_D^2 - 2(x_D-1)x_D y_C^3(y_C^2 + (x_C-1)((2x_C-1)x_D - x_C))y_D + (x_C-1)(x_D-1)^2 x_D^2 y_C^4)/(2(y_C^2 + (x_C-1)^2)y_D(y_D - y_C)(x_C y_D - (x_D-1)y_C)(x_C y_D - x_D y_C)), (y_C(((2x_C-1)y_C^2 + x_C^4 - 2x_C^3 + x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 + (x_C - 1)^2 x_C(2x_D - 1))y_D^3 + y_C^2((2x_D-1)y_C^2 + 2(x_C-1)x_D((3x_C-1)x_D - 4x_C + 1) + 2(x_C-1)x_C)y_D^2 - 2(2x_C-1)(x_D-1)^2 x_D y_C^3 y_D + (x_D-1)^2 x_D^2 y_C^4))/(2(y_C^2 + (x_C - 1)^2)y_D(y_D - y_C)(x_C y_D - (x_D-1)y_C)(x_C y_D - x_D y_C)))$.
The incenter, P, has coordinates
$x_P = \frac{x_C + \sqrt{x_C^2 + y_C^2}}{1 + \sqrt{x_C^2 + y_C^2} + \sqrt{(x_C-1)^2 + y_C^2}}$ and
$y_P = \frac{y_C}{1 + \sqrt{x_C^2 + y_C^2} + \sqrt{(x_C-1)^2 + y_C^2}}$
Line AK is
$y = x(\frac{-x_P}{y_P})$.
Line BK is
$y = (x-1)(\frac{1-x_P}{y_P})$.
Line LM is
$y = y_C + (x - x_C)(\frac{x_P - x_C}{y_C - y_P})$.
K has coordinates $(1 - x_P, (x_P * (x_P - 1))/y_P)$.
L has coordinates
$(-(y_P * (-x_C^2 + x_P * x_C - y_C^2 + y_P * y_C))/(x_C * y_P - x_P * y_C), (x_P * (-x_C^2 + x_P * x_C - y_C^2 + y_P * y_C))/(x_C * y_P - x_P * y_C))$.
M has coordinates
$(-(y_P * (y_C * (y_P - y_C) + (x_C-1) * x_P - x_C^2 + 1) + (x_P - 1) * y_C)/((x_C-1) * y_P - (x_P-1) * y_C), ((x_P-1) * (y_C * (y_P - y_C) + (x_C-1) * (x_P - x_C)))/((x_C-1) * y_P - (x_P-1) * y_C))$.
Line KE is given by the equation

$x(\frac{y_K-y_E}{x_K-x_E}) - y + y_E - x_E(\frac{y_K-y_E}{x_K-x_E}) = 0.$
Line LF is given by the equation
$x(\frac{y_L-y_F}{x_L-x_F}) - y + y_F - x_F(\frac{y_L-y_F}{x_L-x_F}) = 0.$
Line MG is given by the equation
$x(\frac{y_M-y_G}{x_M-x_G}) - y + y_G - x_G(\frac{y_M-y_G}{x_M-x_G}) = 0.$
We calculate that

$$det \begin{bmatrix} \frac{y_K-y_E}{x_K-x_E} & -1 & y_E - x_E(\frac{y_K-y_E}{x_K-x_E}) \\ \frac{y_L-y_F}{x_L-x_F} & -1 & y_F - x_F(\frac{y_L-y_F}{x_L-x_F}) \\ \frac{y_M-y_G}{x_M-x_G} & -1 & y_G - x_G(\frac{y_M-y_G}{x_M-x_G}) \end{bmatrix}$$

$$= -(-2*x_C^7*x_P^2*y_D^3*y_P+2*x_C^7*x_P*y_D^3*y_P+4*x_C^6*x_D*x_P^2*y_C*y_D^2*y_P-4*x_C^6*x_D*x_P*y_C*y_D^2*y_P+2*x_C^6*x_P^3*y_D^3*y_P-2*x_C^6*x_P^2*y_C*y_D^2*y_P+4*x_C^6*x_P^2*y_D^3*y_P-2*x_C^6*x_P*y_C*y_D^3*y_P^2+2*x_C^6*x_P*y_C*y_D^2*y_P-6*x_C^6*x_P*y_D^3*y_P+x_C^6*y_C*y_D^3*y_P^2-2*x_C^5*x_D^2*x_P^2*y_C^2*y_D*y_P+2*x_C^5*x_D^2*x_P*y_C^2*y_D*y_P-4*x_C^5*x_D*x_P^3*y_C*y_D^2*y_P+2*x_C^5*x_D*x_P^2*y_C^2*y_D*y_P-6*x_C^5*x_D*x_P^2*y_C*y_D^2*y_P+4*x_C^5*x_D*x_P*y_C^2*y_D^2*y_P^2-2*x_C^5*x_D*x_P*y_C^2*y_D*y_P+10*x_C^5*x_D*x_P*y_C*y_D^2*y_P-2*x_C^5*x_D*y_C^2*y_D^2*y_P^2+2*x_C^5*x_P^4*y_C*y_D^3-4*x_C^5*x_P^3*y_C*y_D^3+2*x_C^5*x_P^3*y_C*y_D^2*y_P-6*x_C^5*x_P^3*y_D^3*y_P-4*x_C^5*x_P^2*y_C^2*y_D^3*y_P+6*x_C^5*x_P^2*y_C*y_D^3*y_P^2+2*x_C^5*x_P^2*y_C*y_D^3+2*x_C^5*x_P^2*y_C*y_D^2*y_P+4*x_C^5*x_P*y_C^2*y_D^3*y_P-2*x_C^5*x_P*y_C^2*y_D^2*y_P^2-4*x_C^5*x_P*y_C*y_D^2*y_P+6*x_C^5*x_P*y_D^3*y_P+x_C^5*y_C^2*y_D^2*y_P^2-3*x_C^5*y_C*y_D^3*y_P^2+2*x_C^4*x_D^2*x_P^3*y_C^2*y_D*y_P+2*x_C^4*x_D^2*x_P^2*y_C^2*y_D*y_P-2*x_C^4*x_D^2*x_P*y_C^3*y_D*y_P^2-4*x_C^4*x_D^2*x_P*y_C^2*y_D*y_P+x_C^4*x_D^2*y_C^3*y_D*y_P^2-4*x_C^4*x_D*x_P^4*y_C^2*y_D^2+8*x_C^4*x_D*x_P^3*y_C^2*y_D^2-2*x_C^4*x_D*x_P^3*y_C^2*y_D*y_P+10*x_C^4*x_D*x_P^3*y_C*y_D^2*y_P+8*x_C^4*x_D*x_P^2*y_C^3*y_D^2*y_P-12*x_C^4*x_D*x_P^2*y_C^2*y_D^2*y_P^2-4*x_C^4*x_D*x_P^2*y_C^2*y_D^2-2*x_C^4*x_D*x_P^2*y_C^2*y_D*y_P-2*x_C^4*x_D*x_P^2*y_C*y_D^2*y_P-8*x_C^4*x_D*x_P*y_C^3*y_D^2*y_P+2*x_C^4*x_D*x_P*y_C^3*y_D*y_P^2+2*x_C^4*x_D*x_P*y_C^2*y_D^2*y_P^2+4*x_C^4*x_D*x_P*y_C^2*y_D*y_P-8*x_C^4*x_D*x_P*y_C*y_D^2*y_P^2-x_C^4*x_D*y_C^3*y_D*y_P^2+5*x_C^4*x_D*y_C^2*y_D^2*y_P^2-2*x_C^4*x_P^5*y_C*y_D^3+2*x_C^4*x_P^4*y_C^2*y_D^2+4*x_C^4*x_P^3*y_C^2*y_D^3*y_P-4*x_C^4*x_P^3*y_C^2*y_D^2-2*x_C^4*x_P^3*y_C*y_D^3*y_P^2+6*x_C^4*x_P^3*y_C*y_D^3-4*x_C^4*x_P^3*y_C*y_D^2*y_P+6*x_C^4*x_P^3*y_D^3*y_P-4*x_C^4*x_P^2*y_C^3*y_D^2*y_P+4*x_C^4*x_P^2*y_C^2*y_D^3*y_P+6*x_C^4*x_P^2*y_C^2*y_D^2*y_P^2+2*x_C^4*x_P^2*y_C^2*y_D^2-12*x_C^4*x_P^2*y_C*y_D^3*y_P^2-4*x_C^4*x_P^2*y_C*y_D^3+2*x_C^4*x_P^2*y_C*y_D^2*y_P-4*x_C^4*x_P^2*y_D^3*y_P-4*x_C^4*x_P*y_C^3*y_D^3*y_P^2+4*x_C^4*x_P*y_C^3*y_D^2*y_P+4*x_C^4*x_P*y_C^2*y_D^3*y_P^2-8*x_C^4*x_P*y_C^2*y_D^3*y_P-2*x_C^4*x_P*y_C^2*y_D^2*y_P^2+8*x_C^4*x_P*y_C*y_D^3*y_P^2+2*x_C^4*x_P*y_C*y_D^2*y_P-2*x_C^4*x_P*y_D^3*y_P^2+2*x_C^4*y_C^3*y_D^3*y_P^2-2*x_C^4*y_C^2*y_D^3*y_P^2-2*x_C^4*y_C^2*y_D^2*y_P^2+3*x_C^4*y_C*y_D^3*y_P^2+2*x_C^3*x_D^2*x_P^4*y_C^3*y_D-4*x_C^3*x_D^2*x_P^3*y_C^3*y_D-4*x_C^3*x_D^2*x_P^3*y_C^2*y_D*y_P-4*x_C^3*x_D^2*x_P^2*y_C^4*y_D*y_P+6*x_C^3*x_D^2*x_P^2*y_C^3*y_D*y_P^2+2*x_C^3*x_D^2*x_P^2*y_C^3*y_D+2*x_C^3*x_D^2*x_P^2*y_C^2*y_D^2*y_P+4*x_C^3*x_D^2*x_P*y_C^4*y_D*y_P-2*x_C^3*x_D^2*x_P*y_C^3*y_D*y_P^2+2*x_C^3*x_D^2*y_C^3*y_D*y_P^2-2*x_C^3*x_D*x_P^5*y_C^2*y_D^2+2*x_C^3*x_D*x_P^4*y_C^3*y_D^2-2*x_C^3*x_D*x_P^4*y_C^2*y_D^2-8*x_C^3*x_D*x_P^3*y_C^3*y_D^2*y_P+4*x_C^3*x_D*x_P^3*y_C^3*y_D+4*x_C^3*x_D*x_P^3*y_C^2*y_D^2*y_P^2-8*x_C^3*x_D*x_P^3*y_C^2*y_D^2+4*x_C^3*x_D*x_P^3*y_C^2*y_D*y_P-8*x_C^3*x_D*x_P^3*y_C*y_D^2*y_P+4*x_C^3*x_D*x_P^2*y_C^4*y_D*y_P-4*x_C^3*x_D*x_P^2*y_C^3*y_D^2*y_P-6*x_C^3*x_D*x_P^2*y_C^3*y_D*y_P^2-2*x_C^3*x_D*x_P^2*y_C^3*y_D+18*x_C^3*x_D*x_P^2*y_C^2*y_D^2*y_P^2+6*x_C^3*x_D*x_P^2*y_C^2*y_D^2-2*x_C^3*x_D*x_P^2*y_C^2*y_D*y_P+6*x_C^3*x_D*x_P^2*y_C*y_D^2*y_P+8*x_C^3*x_D*x_P*y_C^4*y_D^2*y_P^2-4*x_C^3*x_D*x_P*y_C^4*y_D*y_P-8*x_C^3*x_D*x_P*y_C^3*y_D^2*y_P^2+12*x_C^3*x_D*x_P*y_C^3*y_D^2*y_P+2*x_C^3*x_D*x_P*y_C^3*y_D*y_P^2-14*x_C^3*x_D*x_P*y_C^2*y_D^2*y_P^2-2*x_C^3*x_D*x_P*y_C^2*y_D*y_P+2*x_C^3*x_D*x_P*y_C*y_D^2*y_P-4*x_C^3*x_D*y_C^4*y_D^2*y_P^2+4*x_C^3*x_D*y_C^3*y_D^2*y_P^3+2*x_C^3*x_D*y_C^3*y_D*y_P^2-4*x_C^3*x_D*y_C^2*y_D^2*y_P^2-2*x_C^3*x_P^5*y_C^2*y_D^2+4*x_C^3*x_P^5*y_C*y_D^3+2*x_C^3*x_P^4*y_C^3*y_D^3-4*x_C^3*x_P^4*y_C^2*y_D^3*y_P+2*x_C^3*x_P^4*y_C^2*y_D^2-6*x_C^3*x_P^4*y_C*y_D^3-4*x_C^3*x_P^3*y_C^3*y_D^3+4*x_C^3*x_P^3*y_C^3*y_D^2*y_P-2*x_C^3*x_P^3*y_C^2*y_D^2*y_P^2+2*x_C^3*x_P^3*y_C^2*y_D^2+4*x_C^3*x_P^3*y_C*y_D^3*y_P^2+2*x_C^3*x_P^3*y_C*y_D^2*y_P-2*x_C^3*x_P^3*y_D^3*y_P-2*x_C^3*x_P^2*y_C^4*y_D^3*y_P+6*x_C^3*x_P^2*$$

$$y_C^3*y_D^3*y_P^2+2*x_C^3*x_P^2*y_C^3*y_D^3-4*x_C^3*x_P^2*y_C^2*y_D^3*y_P^3-6*x_C^3*x_P^2*y_C^2*y_D^2*y_P^2-2*x_C^3*x_P^2*y_C^2*y_D^2+6*x_C^3*x_P^2*y_C*y_D^3*y_P^2+2*x_C^3*x_P^2*y_C*y_D^3-2*x_C^3*x_P^2*y_C*y_D^2*y_P+2*x_C^2*x_P^2*y_D^3*y_P+2*x_C^3*x_P*y_C^4*y_D^3*y_P-4*x_C^3*x_P*y_C^4*y_D^2*y_P^2+2*x_C^3*x_P*y_C^3*y_D^3*y_P^2+4*x_C^3*x_P*y_C^3*y_D^2*y_P^3-4*x_C^3*x_P*y_C^3*y_D^2*y_P-4*x_C^3*x_P*y_C^2*y_D^3*y_P^3+4*x_C^3*x_P*y_C^2*y_D^3*y_P+6*x_C^3*x_P*y_C^2*y_D^2*y_P^2-8*x_C^3*x_P*y_C*y_D^3*y_P^2+2*x_C^3*y_C^4*y_D^2*y_P^2-4*x_C^3*y_C^3*y_D^3*y_P^2-2*x_C^3*y_C^3*y_D^2*y_P^3+4*x_C^3*y_C^2*y_D^3*y_P^3+x_C^3*y_C^2*y_D^2*y_P^2-x_C^3*y_C*y_D^3*y_P^2-2*x_C^2*x_D^2*x_P^5*y_C^3*y_D+2*x_C^2*x_D^2*x_P^4*y_C^3*y_D+4*x_C^2*x_D^2*x_P^3*y_C^4*y_D*y_P-2*x_C^2*x_D^2*x_P^3*y_C^3*y_D*y_P^2+2*x_C^2*x_D^2*x_P^3*y_C^3*y_D+2*x_C^2*x_D^2*x_P^3*y_C^2*y_D*y_P-6*x_C^2*x_D^2*x_P^2*y_C^3*y_D*y_P^2-2*x_C^2*x_D^2*x_P^2*y_C^3*y_D-2*x_C^2*x_D^2*x_P^2*y_C^2*y_D*y_P-4*x_C^2*x_D^2*x_P*y_C^5*y_D*y_P^2+4*x_C^2*x_D^2*x_P*y_C^4*y_D*y_P^3-4*x_C^2*x_D^2*x_P*y_C^4*y_D*y_P+6*x_C^2*x_D^2*x_P*y_C^3*y_D*y_P^2+2*x_C^2*x_D^2*y_C^5*y_D*y_P^2-2*x_C^2*x_D^2*y_C^4*y_D*y_P^3+x_C^2*x_D^2*y_C^3*y_D*y_P^2+2*x_C^2*x_D*x_P^5*y_C^3*y_D-6*x_C^2*x_D*x_P^5*y_C^2*y_D^2-4*x_C^2*x_D*x_P^4*y_C^4*y_D^2+8*x_C^2*x_D*x_P^4*y_C^3*y_D^2*y_P-2*x_C^2*x_D*x_P^4*y_C^3*y_D+10*x_C^2*x_D*x_P^4*y_C^2*y_D^2+8*x_C^2*x_D*x_P^3*y_C^4*y_D^2-4*x_C^2*x_D*x_P^3*y_C^4*y_D*y_P-4*x_C^2*x_D*x_P^3*y_C^3*y_D^2*y_P+2*x_C^2*x_D*x_P^3*y_C^3*y_D*y_P^2-2*x_C^2*x_D*x_P^3*y_C^3*y_D-6*x_C^2*x_D*x_P^3*y_C^2*y_D^2*y_P^2-2*x_C^2*x_D*x_P^3*y_C^2*y_D^2-2*x_C^2*x_D*x_P^2*y_C^3*y_D*y_P+2*x_C^2*x_D*x_P^2*y_C*y_D^2*y_P+4*x_C^2*x_D*x_P^2*y_C^5*y_D^2*y_P-12*x_C^2*x_D*x_P^2*y_C^4*y_D^2*y_P^2-4*x_C^2*x_D*x_P^2*y_C^4*y_D^2+8*x_C^2*x_D*x_P^2*y_C^3*y_D^2*y_P^3+6*x_C^2*x_D*x_P^2*y_C^3*y_D*y_P^2+2*x_C^2*x_D*x_P^2*y_C^3*y_D^3-6*x_C^2*x_D*x_P^2*y_C^2*y_D^2*y_P^2-2*x_C^2*x_D*x_P^2*y_C^2*y_D^2+2*x_C^2*x_D*x_P^2*y_C^2*y_D*y_P-2*x_C^2*x_D*x_P^2*y_C*y_D^3*y_P-4*x_C^2*x_D*x_P*y_C^5*y_D^2*y_P+4*x_C^2*x_D*x_P*y_C^5*y_D*y_P^2-4*x_C^2*x_D*x_P*y_C^4*y_D*y_P^3+4*x_C^2*x_D*x_P*y_C^4*y_D*y_P+4*x_C^2*x_D*x_P*y_C^3*y_D^2*y_P^3-4*x_C^2*x_D*x_P*y_C^3*y_D^2*y_P-6*x_C^2*x_D*x_P*y_C^3*y_D*y_P^2+10*x_C^2*x_D*x_P*y_C^2*y_D^2*y_P^2-2*x_C^2*x_D*y_C^5*y_D*y_P^2+6*x_C^2*x_D*y_C^4*y_D^2*y_P^2+2*x_C^2*x_D*y_C^4*y_D*y_P^3-6*x_C^2*x_D*y_C^3*y_D^2*y_P^3-x_C^2*x_D*y_C^3*y_D*y_P^2+x_C^2*x_D*y_C^2*y_D^2*y_P^2+2*x_C^2*x_P^5*y_C^2*y_D^2-2*x_C^2*x_P^5*y_C*y_D^3+2*x_C^2*x_P^4*y_C^4*y_D^2-3*x_C^2*x_P^4*y_C^3*y_D^3-4*x_C^2*x_P^4*y_C^3*y_D^2*y_P+6*x_C^2*x_P^4*y_C^2*y_D^3*y_P-4*x_C^2*x_P^4*y_C^2*y_D^2+4*x_C^2*x_P^4*y_C*y_D^3+2*x_C^2*x_P^3*y_C^4*y_D^3*y_P-4*x_C^2*x_P^3*y_C^4*y_D^2-2*x_C^2*x_P^3*y_C^3*y_D^3*y_P^2+6*x_C^2*x_P^3*y_C^3*y_D^3+4*x_C^2*x_P^3*y_C^3*y_D^2*y_P-8*x_C^2*x_P^3*y_C^2*y_D^3*y_P+2*x_C^2*x_P^3*y_C^2*y_D^2*y_P^2+2*x_C^2*x_P^3*y_C^2*y_D^2-2*x_C^2*x_P^3*y_C*y_D^3*y_P^2-2*x_C^2*x_P^3*y_C*y_D^3-2*x_C^2*x_P^2*y_C^5*y_D^2*y_P+6*x_C^2*x_P^2*y_C^4*y_D^2*y_P^2+2*x_C^2*x_P^2*y_C^4*y_D^2-6*x_C^2*x_P^2*y_C^3*y_D^3*y_P^2-3*x_C^2*x_P^2*y_C^3*y_D^3-4*x_C^2*x_P^2*y_C^3*y_D^2*y_P^3+6*x_C^2*x_P^2*y_C^2*y_D^3*y_P^3+2*x_C^2*x_P^2*y_C^2*y_D^3*y_P-2*x_C^2*x_P*y_C^5*y_D^3*y_P^2+2*x_C^2*x_P*y_C^5*y_D^2*y_P+4*x_C^2*x_P*y_C^4*y_D^3*y_P^3-2*x_C^2*x_P*y_C^4*y_D^3*y_P-2*x_C^2*x_P*y_C^4*y_D^2*y_P^2-2*x_C^2*x_P*y_C^3*y_D^4*y_P^3+4*x_C^2*x_P*y_C^3*y_D^3*y_P^2-2*x_C^2*x_P*y_C^3*y_D^2*y_P^3-2*x_C^2*x_P*y_C^2*y_D^3*y_P^3+2*x_C^2*x_P*y_C*y_D^3*y_P^2+x_C^2*y_C^5*y_D^3*y_P^2-2*x_C^2*y_C^4*y_D^3*y_P^3-2*x_C^2*y_C^4*y_D^2*y_P^2+x_C^2*y_C^3*y_D^3*y_P^4+2*x_C^2*y_C^3*y_D^3*y_P^2+2*x_C^2*y_C^3*y_D^2*y_P^3-2*x_C^2*y_C^2*y_D^3*y_P^3+2*x_C*x_D^2*x_P^5*y_C^3*y_D+2*x_C*x_D^2*x_P^4*y_C^5*y_D-4*x_C*x_D^2*x_P^4*y_C^4*y_D*y_P-4*x_C*x_D^2*x_P^4*y_C^3*y_D-4*x_C*x_D^2*x_P^3*y_C^5*y_D+4*x_C*x_D^2*x_P^3*y_C^4*y_D*y_P+2*x_C*x_D^2*x_P^3*y_C^3*y_D*y_P^2+2*x_C*x_D^2*x_P^3*y_C^3*y_D-2*x_C*x_D^2*x_P^2*y_C^6*y_D*y_P+6*x_C*x_D^2*x_P^2*y_C^5*y_D*y_P^2+2*x_C*x_D^2*x_P^2*y_C^5*y_D-4*x_C*x_D^2*x_P^2*y_C^4*y_D*y_P^3+2*x_C*x_D^2*x_P*y_C^6*y_D*y_P-2*x_C*x_D^2*x_P*y_C^5*y_D*y_P^2-2*x_C*x_D^2*x_P*y_C^3*y_D*y_P^2-2*x_C*x_D^2*y_C^5*y_D*y_P^2+2*x_C*x_D^2*y_C^4*y_D*y_P^3-2*x_C*x_D*x_P^5*y_C^3*y_D+2*x_C*x_D*x_P^5*y_C^2*y_D^2-2*x_C*x_D*x_P^4*y_C^5*y_D+4*x_C*x_D*x_P^4*y_C^4*y_D^2+4*x_C*x_D*x_P^4*y_C^4*y_D*y_P-8*x_C*x_D*x_P^4*y_C^3*y_D^2*y_P+4*x_C*x_D*x_P^4*y_C^3*y_D-4*x_C*x_D*x_P^4*y_C^2*y_D^3-4*x_C*x_D*x_P^3*y_C^5*y_D^2+4*x_C*x_D*x_P^3*y_C^5*y_D-4*x_C*x_D*x_P^3*y_C^4*y_D^2*y_P+4*x_C*x_D*x_P^3*y_C^4*y_D^2+4*x_C*x_D*x_P^3*y_C^3*y_D^3-8*x_C*x_D*x_P^3*y_C^3*y_D^2*y_P+4*x_C*x_D*x_P^3*y_C^3*y_D-4*x_C*x_D*x_P^3*y_C^2*y_D^3*y_P-2*x_C*x_D*x_P^2*y_C^6*y_D^2*y_P+2*x_C*x_D*x_P^2*y_C^6*y_D^2+2*x_C*x_D*x_P^2*y_C^5*y_D^3*y_P-6*x_C*x_D*x_P^2*y_C^5*y_D^2*y_P+2*x_C*x_D*x_P^2*y_C^5*y_D^2-2*x_C*x_D*x_P^2*y_C^4*y_D^3*y_P^2+4*x_C*x_D*x_P^2*y_C^4*y_D^3*y_P-4*x_C*x_D*x_P^2*y_C^4*y_D^2*y_P+4*x_C*x_D*x_P^2*y_C^3*y_D^3*y_P-4*x_C*x_D*x_P^2*y_C^3*y_D^2+4*x_C*x_D*x_P*y_C^6*y_D^2*y_P-4*x_C*x_D*x_P*y_C^6*y_D^2+4*x_C*x_D*x_P*y_C^5*y_D^3*y_P^2-4*x_C*x_D*x_P*y_C^5*y_D^3*y_P+4*x_C*x_D*x_P*y_C^5*y_D^2*y_P^3-4*x_C*x_D*x_P*y_C^5*y_D^2*y_P+4*x_C*x_D*x_P*y_C^4*y_D^3*y_P^3-4*x_C*x_D*x_P*y_C^4*y_D^3*y_P^2+4*x_C*x_D*x_P*y_C^4*y_D^2*y_P^3-4*x_C*x_D*x_P*y_C^3*y_D^3*y_P^3+4*x_C*x_D*x_P*y_C^3*y_D^3*y_P^2-4*x_C*x_D*x_P*y_C^3*y_D^2*y_P^3+4*x_C*x_D*x_P*y_C^2*y_D^3*y_P^3+4*x_C*x_D*y_C^6*y_D^2*y_P^2-4*x_C*x_D*y_C^5*y_D^3*y_P^2-4*x_C*x_D*y_C^5*y_D^2*y_P^3+4*x_C*x_D*y_C^4*y_D^3*y_P^3+x_C*x_P^6*y_C^3*y_D^3*y_P^2-2*x_C*x_P^6*y_C^3*y_D^3*y_P+x_C*x_P^6*y_C^3*y_D^2+2*x_C*x_P^5*y_C^4*y_D^3*y_P-2*x_C*x_P^5*y_C^4*y_D^2-2*x_C*x_P^5*y_C^3*y_D^3*y_P^2+2*x_C*x_P^5*y_C^3*y_D^3*y_P-2*x_C*x_P^5*y_C^3*y_D^2*y_P^3+2*x_C*x_P^5*y_C^3*y_D^2*y_P+2*x_C*x_P^5*y_C^2*y_D^3*y_P^3-2*x_C*x_P^5*y_C^2*y_D^3*y_P+x_C*x_P^4*y_C^6*y_D^3*y_P^2-2*x_C*x_P^4*y_C^6*y_D^3*y_P+x_C*x_P^4*y_C^6*y_D^2+2*x_C*x_P^4*y_C^5*y_D^3*y_P-2*x_C*x_P^4*y_C^5*y_D^2-2*x_C*x_P^4*y_C^4*y_D^3*y_P^2+2*x_C*x_P^4*y_C^4*y_D^3*y_P-2*x_C*x_P^4*y_C^4*y_D^2*y_P^3+2*x_C*x_P^4*y_C^4*y_D^2*y_P+2*x_C*x_P^4*y_C^3*y_D^3*y_P^3-2*x_C*x_P^4*y_C^3*y_D^3*y_P^2+2*x_C*x_P^4*y_C^3*y_D^2*y_P^3-2*x_C*x_P^4*y_C^3*y_D^2*y_P+2*x_C*x_P^4*y_C^2*y_D^3*y_P^3-2*x_C*x_P^4*y_C^2*y_D^3*y_P+x_C*x_P^3*y_C^6*y_D^3*y_P^2-2*x_C*x_P^3*y_C^6*y_D^3*y_P+x_C*x_P^3*y_C^6*y_D^2+2*x_C*x_P^3*y_C^5*y_D^3*y_P-2*x_C*x_P^3*y_C^5*y_D^2-2*x_C*x_P^3*y_C^4*y_D^3*y_P^2+2*x_C*x_P^3*y_C^4*y_D^3*y_P-2*x_C*x_P^3*y_C^4*y_D^2*y_P^3+2*x_C*x_P^3*y_C^4*y_D^2*y_P+2*x_C*x_P^3*y_C^3*y_D^3*y_P^3-2*x_C*x_P^3*y_C^3*y_D^3*y_P^2+2*x_C*x_P^3*y_C^3*y_D^2*y_P^3-2*x_C*x_P^3*y_C^3*y_D^2*y_P+2*x_C*x_P^3*y_C^2*y_D^3*y_P^3-2*x_C*x_P^3*y_C^2*y_D^3*y_P+x_C*x_P^2*y_C^6*y_D^3*y_P^2-2*x_C*x_P^2*y_C^6*y_D^3*y_P+x_C*x_P^2*y_C^6*y_D^2+2*x_C*x_P^2*y_C^5*y_D^3*y_P-2*x_C*x_P^2*y_C^5*y_D^2-2*x_C*x_P^2*y_C^4*y_D^3*y_P^2+2*x_C*x_P^2*y_C^4*y_D^3*y_P-2*x_C*x_P^2*y_C^4*y_D^2*y_P^3+2*x_C*x_P^2*y_C^4*y_D^2*y_P+2*x_C*x_P^2*y_C^3*y_D^3*y_P^3-2*x_C*x_P^2*y_C^3*y_D^3*y_P^2+2*x_C*x_P^2*y_C^3*y_D^2*y_P^3-2*x_C*x_P^2*y_C^3*y_D^2*y_P+2*x_C*x_P^2*y_C^2*y_D^3*y_P^3-2*x_C*x_P^2*y_C^2*y_D^3*y_P$$

$x_C*x_D*x_P*y_C^4*y_D^2*y_P^4-6*x_C*x_D*x_P*y_C^4*y_D^2*y_P^2+4*x_C*x_D*x_P*y_C^3*y_D^2*y_P^3+2*x_C*x_D*x_P*y_C^3*y_D*y_P^2-2*x_C*x_D*x_P*y_C^2*y_D^2*y_P^2-2*x_C*x_D*y_C^6*y_D^2*y_P^2+4*x_C*x_D*y_C^5*y_D^2*y_P^3+2*x_C*x_D*y_C^5*y_D*y_P^2-2*x_C*x_D*y_C^4*y_D^2*y_P^4-2*x_C*x_D*y_C^4*y_D^2*y_P^2-2*x_C*x_D*y_C^4*y_D*y_P^3+2*x_C*x_D*y_C^3*y_D^2*y_P^3-x_C*x_P^4*y_C^4*y_D^2+x_C*x_P^4*y_C^3*y_D^3+2*x_C*x_P^4*y_C^3*y_D^2*y_P-2*x_C*x_P^4*y_C^2*y_D^3*y_P+2*x_C*x_P^3*y_C^5*y_D^2*y_P-2*x_C*x_P^3*y_C^4*y_D^3*y_P-2*x_C*x_P^3*y_C^4*y_D^2*y_P^2+2*x_C*x_P^3*y_C^4*y_D^2*y_P+2*x_C*x_P^3*y_C^3*y_D^3*y_P^2-2*x_C*x_P^3*y_C^3*y_D^3-4*x_C*x_P^3*y_C^3*y_D^2*y_P+4*x_C*x_P^3*y_C^2*y_D^3*y_P-2*x_C*x_P^2*y_C^5*y_D^2*y_P+2*x_C*x_P^2*y_C^4*y_D^3*y_P-x_C*x_P^2*y_C^4*y_D^2+x_C*x_P^2*y_C^3*y_D^3+2*x_C*x_P^2*y_C^3*y_D^2*y_P^2+2*x_C*x_P^2*y_C^3*y_D^2*y_P-2*x_C*x_P^2*y_C^2*y_D^3*y_P^3-2*x_C*x_P^2*y_C^2*y_D^3*y_P-2*x_C*x_P*y_C^6*y_D^2*y_P^2+2*x_C*x_P*y_C^5*y_D^3*y_P^2+4*x_C*x_P*y_C^5*y_D^2*y_P^3-4*x_C*x_P*y_C^4*y_D^3*y_P^3-2*x_C*x_P*y_C^4*y_D^2*y_P^4+2*x_C*x_P*y_C^4*y_D^2*y_P^2+2*x_C*x_P*y_C^3*y_D^3*y_P^4-2*x_C*x_P*y_C^3*y_D^3*y_P^2-2*x_C*x_P*y_C^3*y_D^2*y_P^3+2*x_C*x_P*y_C^2*y_D^3*y_P^3+x_C*y_C^6*y_D^2*y_P^2-x_C*y_C^5*y_D^3*y_P^2-2*x_C*y_C^5*y_D^2*y_P^3+2*x_C*y_C^4*y_D^3*y_P^3+x_C*y_C^4*y_D^2*y_P^4-x_C*y_C^3*y_D^3*y_P^4-x_D^2*x_P^4*y_C^5*y_D+2*x_D^2*x_P^4*y_C^4*y_D*y_P+2*x_D^2*x_P^3*y_C^6*y_D*y_P-2*x_D^2*x_P^3*y_C^5*y_D*y_P^2+2*x_D^2*x_P^3*y_C^5*y_D-4*x_D^2*x_P^3*y_C^4*y_D*y_P-2*x_D^2*x_P^2*y_C^6*y_D*y_P-x_D^2*x_P^2*y_C^5*y_D+2*x_D^2*x_P^2*y_C^4*y_D*y_P^3+2*x_D^2*x_P^2*y_C^4*y_D*y_P-2*x_D^2*x_P*y_C^7*y_D*y_P^2+4*x_D^2*x_P*y_C^6*y_D*y_P^3-2*x_D^2*x_P*y_C^5*y_D*y_P^4+2*x_D^2*x_P*y_C^5*y_D*y_P^2-2*x_D^2*x_P*y_C^4*y_D*y_P^3+x_D^2*y_C^7*y_D*y_P^2-2*x_D^2*y_C^6*y_D*y_P^3+x_D^2*y_C^5*y_D*y_P^4+x_D*x_P^4*y_C^5*y_D-x_D*x_P^4*y_C^4*y_D^2-2*x_D*x_P^4*y_C^4*y_D*y_P+2*x_D*x_P^4*y_C^3*y_D^2*y_P-2*x_D*x_P^3*y_C^6*y_D*y_P+2*x_D*x_P^3*y_C^5*y_D^2*y_P+2*x_D*x_P^3*y_C^5*y_D*y_P^2-2*x_D*x_P^3*y_C^5*y_D-2*x_D*x_P^3*y_C^4*y_D^2*y_P^2+2*x_D*x_P^3*y_C^4*y_D^2+4*x_D*x_P^3*y_C^4*y_D*y_P-4*x_D*x_P^3*y_C^3*y_D^2*y_P+2*x_D*x_P^2*y_C^6*y_D*y_P-2*x_D*x_P^2*y_C^5*y_D^2*y_P+x_D*x_P^2*y_C^5*y_D-x_D*x_P^2*y_C^4*y_D^2-2*x_D*x_P^2*y_C^4*y_D*y_P^3-2*x_D*x_P^2*y_C^4*y_D*y_P+2*x_D*x_P^2*y_C^3*y_D^2*y_P^3+2*x_D*x_P^2*y_C^3*y_D^2*y_P+2*x_D*x_P*y_C^7*y_D*y_P^2-2*x_D*x_P*y_C^6*y_D^2*y_P^2-4*x_D*x_P*y_C^6*y_D*y_P^3+4*x_D*x_P*y_C^5*y_D^2*y_P^3+2*x_D*x_P*y_C^5*y_D*y_P^4-2*x_D*x_P*y_C^5*y_D*y_P^2-2*x_D*x_P*y_C^4*y_D^2*y_P^4+2*x_D*x_P*y_C^4*y_D^2*y_P^2+2*x_D*x_P*y_C^4*y_D*y_P^3-2*x_D*x_P*y_C^3*y_D^2*y_P^3-x_D*y_C^7*y_D*y_P^2+x_D*y_C^6*y_D^2*y_P^2+2*x_D*y_C^6*y_D*y_P^3-2*x_D*y_C^5*y_D^2*y_P^3-x_D*y_C^5*y_D*y_P^4+x_D*y_C^4*y_D^2*y_P^4)/(y_P*(y_C-y_D+x_C*y_D-x_D*y_C-x_P*y_C+x_P*y_D)*(y_C^2*y_P^2-y_C^3*y_P-x_C^2*y_C*y_P+x_C^2*y_D*y_P+x_D*y_C^3*y_P-x_C^3*y_D*y_P-x_D*y_C^2*y_P^2-x_C*x_P*y_C*y_D+x_C*x_P*y_C*y_P+x_C^2*x_D*y_C*y_P+x_C^2*x_P*y_D*y_P+x_C*y_C*y_D*y_P^2-x_C*y_C^2*y_D*y_P-x_C*x_D*x_P*y_C*y_P)*(y_C*y_D-y_D*y_P+x_C^2*y_D*y_P+x_D*y_C^3*y_P-x_C^3*y_D*y_P-y_C*y_D*y_P^2+y_C^2*y_D*y_P-x_D*y_C^2*y_P^2-x_C*y_C*y_D+x_C*y_D*y_P-x_P*y_C*y_D+x_P*y_D*y_P-x_C*x_D*y_C*y_P+x_C*x_P*y_C*y_D-2*x_C*x_P*y_D*y_P+x_D*x_P*y_C*y_P+x_C^2*x_D*y_C*y_P+x_C^2*x_P*y_D*y_P+x_C*y_C*y_D*y_P^2-x_C*y_C^2*y_D*y_P-x_C*x_D*x_P*y_C*y_P))$.
We substitute the values of x_P and y_P into the above expression and one can verify that it evaluates to 0 as desired. Thus, we've shown that lines KE, LF, and MG are concurrent.
Line KH is given by the equation
$x(\frac{y_H-y_K}{x_H-x_K})-y+y_K-x_K(\frac{y_H-y_K}{x_H-x_K})=0$.
Line LI is given by the equation
$x(\frac{y_I-y_L}{x_I-x_L})-y+y_L-x_L(\frac{y_I-y_L}{x_I-x_L})=0$.
Line MJ is given by the equation
$x(\frac{y_J-y_M}{x_J-x_M})-y+y_M-x_M(\frac{y_J-y_M}{x_J-x_M})=0$.
We remark that since D is arbitrary, the above result holds for any three concurrent cevians. In [2], we showed that lines AJ, BI, and CH are in fact concurrent cevians. This means, by combining that result with the above, we have that lines KH, LI, and MJ are concurrent as desired. □

The point N becomes the mittenpunkt when D is taken to be the centroid.

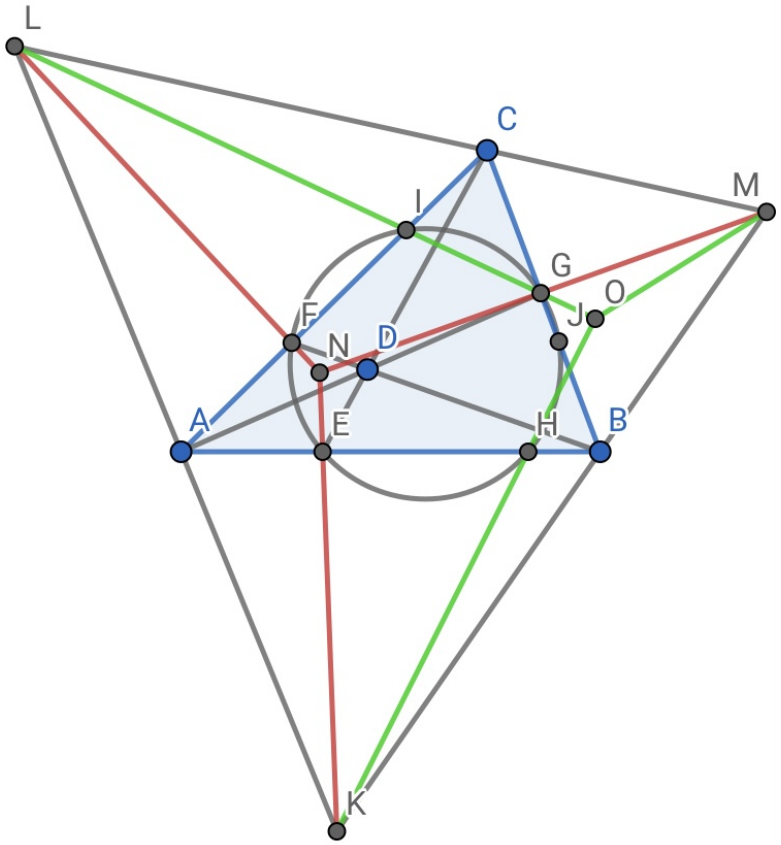

FIGURE 1. Illustration.

REFERENCES

[1] Weisstein, Eric W. "Mittenpunkt." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Mittenpunkt.html

[2] Warren, B.L., "Generalization of the Gergonne Point", *International Journal of Computer Discovered Mathematics*, Vol. 9 (2024), paper 25, pp. 191-196.