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Three Lines Associated to Non-concurrent Cevians

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1. Introduction

Many results exist regarding concurrent cevians of a given triangle, such as Ceva's theorem [1]. Here, we will demonstrate the existence of a particular result related to cevians which are not concurrent.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point on line AB.
Let E be a point on line AC.
Let F be a point on line BC.
Let lines AF and CD intersect at point G.
Let lines BE and CD intersect at point H.
Let lines AF and BE intersect at point I.
Let J be the midpoint of A and H.
Let K be the midpoint of E and G.
Let L be the midpoint of B and G.
Let M be the midpoint of D and I.
Let N be the midpoint of C and I.
Let O be the midpoint of H and F.
Then,
1. the triples of points $(J, K, N), (J, L, M)$, and (L, N, O) are each collinear.

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2. the triples of lines $(CD, JO, KL), (AF, KL, MN)$, *and* (BE, JO, MN) *are each concurrent.*

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates $(x_D, 0)$.
E has coordinates $(x_E, \frac{x_E y_C}{x_C})$.
F has coordinates $(x_F, (x_F - 1)(\frac{y_C}{x_C - 1}))$.
G has coordinates
$(((x_C - 1) * x_D * x_F)/((x_D - 1) * x_F - x_D + x_C), (x_D * y_C * (x_F - 1))/(x_C - x_D - x_F + x_D * x_F))$.
H has coordinates
$((x_C * x_D + x_C * x_E - x_D * x_E - x_C * x_D * x_E)/(x_C - x_D * x_E), (x_E * y_C - x_D * x_E * y_C)/(x_C - x_D * x_E))$.
I has coordinates
$(-(x_E * x_F * (x_C - 1))/(x_C - x_C * x_E - x_C * x_F + x_E * x_F), -(x_E * y_C * (x_F - 1))/(x_C - x_C * x_E - x_C * x_F + x_E * x_F))$.
J has coordinates
$((1/2) * ((x_C * x_D + x_C * x_E - x_D * x_E - x_C * x_D * x_E)/(x_C - x_D * x_E)), (1/2) * ((x_E * y_C - x_D * x_E * y_C)/(x_C - x_D * x_E)))$.
K has coordinates
$((1/2) * (x_E + ((x_C - 1) * x_D * x_F)/((x_D - 1) * x_F - x_D + x_C)), (1/2) * (\frac{x_E y_C}{x_C} + (x_D * y_C * (x_F - 1))/(x_C - x_D - x_F + x_D * x_F)))$.
L has coordinates
$((1/2) * (((x_C - 1) * x_D * x_F)/((x_D - 1) * x_F - x_D + x_C) + 1), (1/2) * ((x_D * y_C * (x_F - 1))/(x_C - x_D - x_F + x_D * x_F)))$.
M has coordinates
$((1/2) * (x_D + (-(x_E * x_F * (x_C - 1))/(x_C - x_C * x_E - x_C * x_F + x_E * x_F))), (1/2) * (-(x_E * y_C * (x_F - 1))/(x_C - x_C * x_E - x_C * x_F + x_E * x_F)))$.
N has coordinates
$((1/2) * (x_C + (-(x_E * x_F * (x_C - 1))/(x_C - x_C * x_E - x_C * x_F + x_E * x_F))), (1/2) * (y_C + (-(x_E * y_C * (x_F - 1))/(x_C - x_C * x_E - x_C * x_F + x_E * x_F))))$.
O has coordinates
$((1/2) * ((x_C * x_D + x_C * x_E - x_D * x_E - x_C * x_D * x_E)/(x_C - x_D * x_E) + x_F), (1/2) * ((x_E * y_C - x_D * x_E * y_C)/(x_C - x_D * x_E) + (x_F - 1)(\frac{y_C}{x_C - 1})))$.

Line JK is given by the equation
$y = (((x_E * y_C - x_D * x_E * y_C)/(2 * (x_C - x_D * x_E)) - ((x_D * (x_F - 1) * y_C)/(x_D * x_F - x_F - x_D + x_C) + (x_E * y_C)/x_C)/2) * (x - (-x_C * x_D * x_E - x_D * x_E + x_C * x_E + x_C * x_D)/(2 * (x_C - x_D * x_E))))/((-x_C * x_D * x_E - x_D * x_E + x_C * x_E + x_C * x_D)/(2 * (x_C - x_D * x_E)) - (((x_C - 1) * x_D * x_F)/((x_D - 1) * x_F - x_D + x_C) + x_E)/2) + (x_E * y_C - x_D * x_E * y_C)/(2 * (x_C - x_D * x_E))$.
We verify that N lies on line JK since we calculate
$(((x_E * y_C - x_D * x_E * y_C)/(2 * (x_C - x_D * x_E)) - ((x_D * (x_F - 1) * y_C)/(x_D * x_F - x_F - x_D + x_C) + (x_E * y_C)/x_C)/2) * (((1/2)(x_C + (-(x_E * x_F * (x_C - 1))/(x_C - x_C * x_E - x_C * x_F + x_E * x_F)))) - (-x_C * x_D * x_E - x_D * x_E + x_C * x_E + x_C * x_D)/(2 * (x_C - x_D * x_E))))/((-x_C * x_D * x_E - x_D * x_E + x_C * x_E + x_C * x_D)/(2 * (x_C - x_D * x_E)) - (((x_C - 1) *$

$x_D*x_F)/((x_D-1)*x_F-x_D+x_C)+x_E)/2)+(x_E*y_C-x_D*x_E*y_C)/(2*(x_C-x_D*x_E))-((1/2)(y_C+(-(x_E*y_C*(x_F-1))/(x_C-x_C*x_E-x_C*x_F+x_E*x_F))))=0$
as desired.
Line JL is given by the equation
$y=(((x_E*y_C-x_D*x_E*y_C)/(2*(x_C-x_D*x_E))-(x_D*(x_F-1)*y_C)/(2*(x_D*x_F-x_F-x_D+x_C)))*(x-(-x_C*x_D*x_E-x_D*x_E+x_C*x_E+x_C*x_D)/(2*(x_C-x_D*x_E))))/((-x_C*x_D*x_E-x_D*x_E+x_C*x_E+x_C*x_D)/(2*(x_C-x_D*x_E))-(((x_C-1)*x_D*x_F)/((x_D-1)*x_F-x_D+x_C)+1)/2)+(x_E*y_C-x_D*x_E*y_C)/(2*(x_C-x_D*x_E))$.
We verify that M lies on line JL since we calculate
$(((x_E*y_C-x_D*x_E*y_C)/(2*(x_C-x_D*x_E))-(x_D*(x_F-1)*y_C)/(2*(x_D*x_F-x_F-x_D+x_C)))*(((1/2)(x_D+(-(x_E*x_F*(x_C-1))/(x_C-x_C*x_E-x_C*x_F+x_E*x_F))))-(-x_C*x_D*x_E-x_D*x_E+x_C*x_E+x_C*x_D)/(2*(x_C-x_D*x_E))))/((-x_C*x_D*x_E-x_D*x_E+x_C*x_E+x_C*x_D)/(2*(x_C-x_D*x_E))-(((x_C-1)*x_D*x_F)/((x_D-1)*x_F-x_D+x_C)+1)/2)+(x_E*y_C-x_D*x_E*y_C)/(2*(x_C-x_D*x_E))-((1/2)(-(x_E*y_C*(x_F-1))/(x_C-x_C*x_E-x_C*x_F+x_E*x_F)))=0$
as desired.
Line LN is given by the equation
$y=(((y_C-(x_E*(x_F-1)*y_C)/(x_E*x_F-x_C*x_F-x_C*x_E+x_C))/2-(x_D*(x_F-1)*y_C)/(2*(x_D*x_F-x_F-x_D+x_C)))*(x-(((x_C-1)*x_D*x_F)/((x_D-1)*x_F-x_D+x_C)+1)/2))/((x_C-((x_C-1)*x_E*x_F)/(x_E*x_F-x_C*x_F-x_C*x_E+x_C))/2-(((x_C-1)*x_D*x_F)/((x_D-1)*x_F-x_D+x_C)+1)/2)+(x_D*(x_F-1)*y_C)/(2*(x_D*x_F-x_F-x_D+x_C))$.
We verify that O lies on line LN since we calculate
$(((y_C-(x_E*(x_F-1)*y_C)/(x_E*x_F-x_C*x_F-x_C*x_E+x_C))/2-(x_D*(x_F-1)*y_C)/(2*(x_D*x_F-x_F-x_D+x_C)))*(((1/2)((x_C*x_D+x_C*x_E-x_D*x_E-x_C*x_D*x_E)/(x_C-x_D*x_E)+x_F))-(((x_C-1)*x_D*x_F)/((x_D-1)*x_F-x_D+x_C)+1)/2))/((x_C-((x_C-1)*x_E*x_F)/(x_E*x_F-x_C*x_F-x_C*x_E+x_C))/2-(((x_C-1)*x_D*x_F)/((x_D-1)*x_F-x_D+x_C)+1)/2)+(x_D*(x_F-1)*y_C)/(2*(x_D*x_F-x_F-x_D+x_C))-((1/2)((x_E*y_C-x_D*x_E*y_C)/(x_C-x_D*x_E)+(x_F-1)(\frac{y_C}{x_C-1})))=0$
as desired.
Line CD is given by the equation
$x(\frac{y_C}{x_C-x_D})-y+y_C-x_C(\frac{y_C}{x_C-x_D})=0.$
Line JO is given by the equation
$x(\frac{y_J-y_O}{x_J-x_O})-y+y_J-x_J(\frac{y_J-y_O}{x_J-x_O})=0.$
Line KL is given by the equation
$x(\frac{y_K-y_L}{x_K-x_L})-y+y_K-x_K(\frac{y_K-y_L}{x_K-x_L})=0.$
Line AF is given by the equation
$x(\frac{y_F}{x_F})-y=0.$
Line MN is given by the equation
$x(\frac{y_M-y_N}{x_M-x_N})-y+y_M-x_M(\frac{y_M-y_N}{x_M-x_N})=0.$
Line BE is given by the equation
$x(\frac{-y_E}{1-x_E})-y-\frac{y_B-y_E}{x_B-x_E}=0.$
We then verify that the triples of points $(J,K,N),(J,L,M)$, and (L,N,O) are each collinear by substituting the above values into the following matrices and calculating that

$$det\begin{bmatrix}\frac{y_C}{x_C-x_D} & -1 & y_C-x_C(\frac{y_C}{x_C-x_D})\\ \frac{y_J-y_O}{x_J-x_O} & -1 & y_J-x_J(\frac{y_J-y_O}{x_J-x_O})\\ \frac{y_K-y_L}{x_K-x_L} & -1 & y_K-x_K(\frac{y_K-y_L}{x_K-x_L})\end{bmatrix}=$$

$$det\begin{bmatrix} \frac{y_F}{x_F} & -1 & 0 \\ \frac{y_K-y_L}{x_K-x_L} & -1 & y_K - x_K(\frac{y_K-y_L}{x_K-x_L}) \\ \frac{y_M-y_N}{x_M-x_N} & -1 & y_M - x_M(\frac{y_M-y_N}{x_M-x_N}) \end{bmatrix} =$$
$$det\begin{bmatrix} \frac{-y_E}{1-x_E} & -1 & -\frac{-y_E}{1-x_E} \\ \frac{y_J-y_O}{x_J-x_O} & -1 & y_J - x_J(\frac{y_J-y_O}{x_J-x_O}) \\ \frac{y_M-y_N}{x_M-x_N} & -1 & y_M - x_M(\frac{y_M-y_N}{x_M-x_N}) \end{bmatrix} = 0$$

as desired.

One can verify even further that these points of concurrency are the midpoints of ΔGHI. □

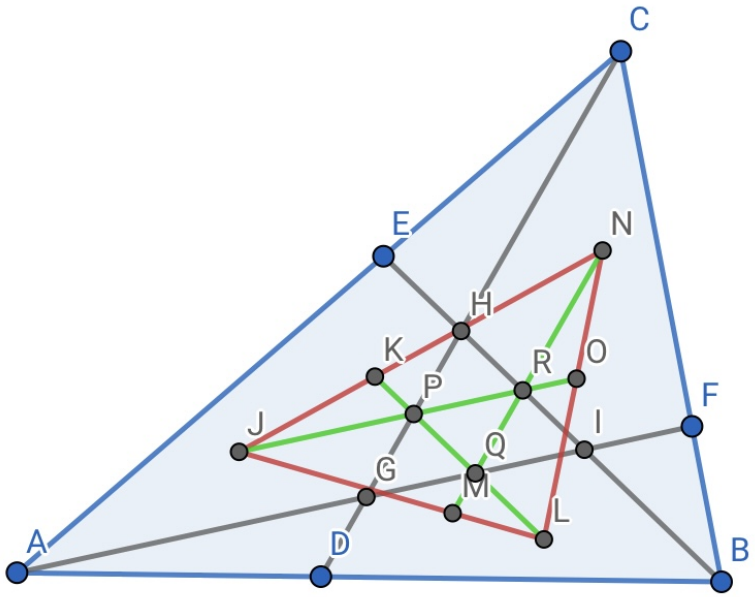

FIGURE 1. The triples of points $(J, K, N), (J, L, M)$, and (L, N, O) are each collinear and the triples of lines $(CD, JO, KL), (AF, KL, MN)$, and (BE, JO, MN) are each concurrent.

REFERENCES

[1] Weisstein, Eric W. "Ceva's Theorem." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/CevasTheorem.html