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Generalization of the Exeter Point

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1. Introduction

The Exeter point [1] is a triangle center discovered in 1986. It is denoted $X(22)$ in Clark Kimberling's Encyclopedia of Triangle Centers. Part of the construction involves the medians of a triangle. Here, we will use some motivation from [2] to generalize this point.

2. Result

Theorem 2.1. *Let A, B, and C denote the vertices of a given triangle.*
Let D be any point in the plane.
Let E be a point on line AD.
Let F be a point on line BD such that $BF : DF = AE : DE$.
Let G be a point on line CD such that $CG : DG = AE : DE$.
Let H denote the midpoint of E and F.
Let I denote the midpoint of E and G.
Let J denote the midpoint of F and G.
Let lines AJ, BI, and CH intersect at point K.
Let line CH intersect the circumcircle of A, B, and C at point L.
Let line AJ intersect the circumcircle of A, B, and C at point M.
Let line BI intersect the circumcircle of A, B, and C at point N.
Let the tangent line through A meet the tangent line through B at point O.
Let the tangent line through B meet the tangent line through C at point P.

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Let the tangent line through A meet the tangent line through C at point Q. Then lines LO, MP, and NQ are concurrent.

Proof. Without loss of generality, we will work in Cartesian coordinates.
The coordinates of A are $(0, 0)$.
The coordinates of B are $(1, 0)$.
The coordinates of C are (x_C, y_C).
The coordinates of D are (x_D, y_D).
The coordinates of E are $(x_E, \frac{x_E y_D}{x_D})$.
The coordinates of F are $(x_E - \frac{x_E}{x_D} + 1, \frac{x_E y_D}{x_D})$.
The coordinates of G are $(x_C + (\frac{x_E}{x_D})(x_D - x_C), y_C + (\frac{x_E}{x_D})(y_D - y_C))$.
The coordinates of H are $(\frac{1}{2}(2x_E - \frac{x_E}{x_D} + 1), \frac{x_E y_D}{x_D})$.
I has coordinates $(\frac{1}{2}(x_E + x_C + (\frac{x_E}{x_D})(x_D - x_C)), \frac{1}{2}(\frac{2x_E y_D + y_C(x_D - x_E)}{x_D}))$.
J has coordinates $(\frac{1}{2}(x_E - \frac{x_E}{x_D} + 1 + x_C + (\frac{x_E}{x_D})(x_D - x_C)), \frac{1}{2}(\frac{2x_E y_D + y_C(x_D - x_E)}{x_D}))$.
K has coordinates $(\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E}, \frac{x_D y_C - x_E y_C + 2x_E y_D}{3x_D - x_E})$.
The circumcircle of A, B, and C is given by the equation
$(x - \frac{1}{2})^2 + (y - \frac{x_C(x_C - 1) + y_C^2}{2y_C})^2 = \frac{1}{4} + (\frac{x_C(x_C - 1) + y_C^2}{2y_C})^2$.
L has coordinates
$(((2x_C x_E y_D - ((2x_D - 1)x_E + x_D)y_C)(2x_E y_C y_D - 2x_D y_C^2 + (x_C - 1)(2x_D - 1)x_E - (x_C - 1)(2x_C - 1)x_D))/(y_C(4x_E^2 y_D^2 - 8x_D x_E y_C y_D + 4x_D^2 y_C^2 + (2x_D - 1)^2 x_E^2 - 2(2x_C - 1)x_D(2x_D - 1)x_E + (2x_C - 1)^2 x_D^2)), ((2(x_C - 1)x_E y_D - ((2x_D - 1)x_E - x_D)y_C)(2x_C x_E y_D - ((2x_D - 1)x_E + x_D)y_C))/(y_C(4x_E^2 y_D^2 - 8x_D x_E y_C y_D + 4x_D^2 y_C^2 + (2x_D - 1)^2 x_E^2 - 2(2x_C - 1)x_D(2x_D - 1)x_E + (2x_C - 1)^2 x_D^2)))$.
M has coordinates
$((((2x_D - x_C - 1)x_E + (x_C + 1)x_D)(2x_E(y_C^2 + (x_C - 1)x_C)y_D - (x_E - x_D)y_C^3 + ((2x_D - x_C^2 - 1)x_E + (x_C^2 + 1)x_D)y_C))/(y_C(4x_E^2 y_D^2 - 4x_E(x_E - x_D)y_C y_D + (x_E - x_D)^2 y_C^2 + (2x_D - x_C - 1)^2 x_E^2 + 2(x_C + 1)x_D(2x_D - x_C - 1)x_E + (x_C + 1)^2 x_D^2)), ((2x_E y_D - (x_E - x_D)y_C)(2x_E(y_C^2 + (x_C - 1)x_C)y_D - (x_E - x_D)y_C^3 + ((2x_D - x_C^2 - 1)x_E + (x_C^2 + 1)x_D)y_C))/(y_C(4x_E^2 y_D^2 - 4x_E(x_E - x_D)y_C y_D + (x_E - x_D)^2 y_C^2 + (2x_D - x_C - 1)^2 x_E^2 + 2(x_C + 1)x_D(2x_D - x_C - 1)x_E + (x_C + 1)^2 x_D^2)))$.
N has coordinates
$(((2x_E y_D - (x_E - x_D)y_C)(2x_E y_C y_D + ((2x_D - x_C - 1)x_E + (x_C - 1)x_D)y_C^2 + (x_C - 1)x_C(2x_D - x_C)x_E + (x_C - 2)(x_C - 1)x_C x_D))/(y_C(4x_E^2 y_D^2 - 4x_E(x_E - x_D)y_C y_D + (x_E - x_D)^2 y_C^2 + (2x_D - x_C)^2 x_E^2 + 2(x_C - 2)x_D(2x_D - x_C)x_E + (x_C - 2)^2 x_D^2)), ((2x_E y_D - (x_E - x_D)y_C)(2x_E(y_C^2 + (x_C - 1)x_C)y_D - (x_E - x_D)y_C^3 - (2x_D x_E + x_C^2 x_E - 2x_C x_E - x_C^2 x_D + 2x_C x_D - 2x_D)y_C))/(y_C(4x_E^2 y_D^2 - 4x_E(x_E - x_D)y_C y_D + (x_E - x_D)^2 y_C^2 + (2x_D - x_C)^2 x_E^2 + 2(x_C - 2)x_D(2x_D - x_C)x_E + (x_C - 2)^2 x_D^2)))$.
The tangent line through A is given by the equation $y = -x(\frac{y_C}{y_C^2 + x_C(x_C - 1)})$.
The tangent line through B is given by the equation $y = (x - 1)(\frac{y_C}{y_C^2 + x_C(x_C - 1)})$.
The tangent line through C is given by the equation $y = y_C + (x - x_C)(-\frac{\frac{1}{2} - x_C}{\frac{x_C(x_C - 1) + y_C^2}{2y_C} - y_C})$.
O has coordinates $(\frac{1}{2}, -\frac{y_C}{2x_C^2 - 2x_C + 2y_C^2})$.
P has coordinates $(\frac{x_C^2 + x_C + y_C^2}{2x_C}, \frac{y_C}{2x_C})$.
Q has coordinates $(\frac{x_C^2 - x_C + y_C^2}{2(x_C - 1)}, -\frac{y_C}{2(x_C - 1)})$.
Line LO is given by the equation
$y = ((-((2(x_C - 1)x_E y_D - ((2x_D - 1)x_E - x_D)y_C)(2x_C x_E y_D - ((2x_D - 1)x_E + x_D)y_C))/(y_C(4x_E^2 y_D^2 - 8x_D x_E y_C y_D + 4x_D^2 y_C^2 + (2x_D - 1)^2 x_E^2 - 2(2x_C - 1)x_D(2x_D -$

$1)x_E+(2x_C-1)^2x_D^2))-y_C/(2y_C^2+2x_C^2-2x_C))(x-((2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C)(2x_Ey_Cy_D-2x_Dy_C^2+(x_C-1)(2x_D-1)x_E+(1-x_C)(2x_C-1)x_D))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2))))/(1/2-((2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C)(2x_Ey_Cy_D-2x_Dy_C^2+(x_C-1)(2x_D-1)x_E+(1-x_C)(2x_C-1)x_D))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2)))+((2(x_C-1)x_Ey_D-((2x_D-1)x_E-x_D)y_C)(2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2)).$

Line MP is given by the equation

$y=((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))(x-(y_C^2+x_C^2+x_C)/(2x_C)))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))+y_C/(2x_C).$

Line NQ is given by the equation

$y=((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))(x-(y_C^2+x_C^2-x_C)/(2(x_C-1))))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))-y_C/(2(x_C-1)).$

Lines MP and NQ intersect at point R which has coordinates

$x_R=(((y_C^2+x_C^2+x_C)/(2x_C))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))]-y_C/(2x_C)-((y_C^2+x_C^2-x_C)/(2(x_C-1)))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))]-y_C/(2(x_C-1)))/((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))-((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+$

$(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))))$
and
$y_R=((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))((((y_C^2+x_C^2+x_C)/(2x_C))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))]-y_C/(2x_C)-((y_C^2+x_C^2-x_C)/(2(x_C-1)))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))]-y_C/(2(x_C-1)))/((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))-((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))))-(y_C^2+x_C^2-x_C)/(2(x_C-1))))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))-y_C/(2(x_C-1))).$
We then verify that R lies on line LO since we calculate that
$((-((2(x_C-1)x_Ey_D-((2x_D-1)x_E-x_D)y_C)(2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2))-y_C/(2y_C^2+2x_C^2-2x_C))(x_R-((2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C)(2x_Ey_Cy_D-2x_Dy_C^2+(x_C-1)(2x_D-1)x_E+(1-x_C)(2x_C-1)x_D))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2))))/(1/2-((2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C)(2x_Ey_Cy_D-2x_Dy_C^2+(x_C-1)(2x_D-1)x_E+(1-x_C)(2x_C-1)x_D))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2)))+((2(x_C-1)x_Ey_D-((2x_D-1)x_E-x_D)y_C)(2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2))-y_R$
$=((-((2(x_C-1)x_Ey_D-((2x_D-1)x_E-x_D)y_C)(2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-$

$$1)x_E+(2x_C-1)^2x_D^2))-y_C/(2y_C^2+2x_C^2-2x_C))((((y_C^2+x_C^2+x_C)/(2x_C))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))]-y_C/(2x_C)-((y_C^2+x_C^2-x_C)/(2(x_C-1)))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))]-y_C/(2(x_C-1)))/((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))-((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))))-((2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C)(2x_Ey_Cy_D-2x_Dy_C^2+(x_C-1)(2x_D-1)x_E+(1-x_C)(2x_C-1)x_D))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2))))/(1/2-((2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C)(2x_Ey_Cy_D-2x_Dy_C^2+(x_C-1)(2x_D-1)x_E+(1-x_C)(2x_C-1)x_D))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2)))+((2(x_C-1)x_Ey_D-((2x_D-1)x_E-x_D)y_C)(2x_Cx_Ey_D-((2x_D-1)x_E+x_D)y_C))/(y_C(4x_E^2y_D^2-8x_Dx_Ey_Cy_D+4x_D^2y_C^2+(2x_D-1)^2x_E^2-2(2x_C-1)x_D(2x_D-1)x_E+(2x_C-1)^2x_D^2))-((((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))(((y_C^2+x_C^2+x_C)/(2x_C))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))]-y_C/(2x_C)-((y_C^2+x_C^2-x_C)/(2(x_C-1)))*[(((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+$$

$(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))]-y_C/(2(x_C-1)))/((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-y_C/(2x_C))/((((2x_D-x_C-1)x_E+(x_C+1)x_D)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+((2x_D-x_C^2-1)x_E+(x_C^2+1)x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C-1)^2x_E^2+2(x_C+1)x_D(2x_D-x_C-1)x_E+(x_C+1)^2x_D^2))-(y_C^2+x_C^2+x_C)/(2x_C))-((((2x_Ey_D-(x_E-x_D)y_C)(2x_E(y_C^2+(x_C-1)x_C)y_D-(x_E-x_D)y_C^3+(-2x_Dx_E-x_C^2x_E+2x_Cx_E+x_C^2x_D-2x_Cx_D+2x_D)y_C))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))+y_C/(2(x_C-1)))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))))-(y_C^2+x_C^2-x_C)/(2(x_C-1))))/(((2x_Ey_D-(x_E-x_D)y_C)(2x_Ey_Cy_D+((2x_D-x_C-1)x_E+(x_C-1)x_D)y_C^2+(x_C-1)x_C(2x_D-x_C)x_E+(x_C-2)(x_C-1)x_Cx_D))/(y_C(4x_E^2y_D^2-4x_E(x_E-x_D)y_Cy_D+(x_E-x_D)^2y_C^2+(2x_D-x_C)^2x_E^2+2(x_C-2)x_D(2x_D-x_C)x_E+(x_C-2)^2x_D^2))-(y_C^2+x_C^2-x_C)/(2(x_C-1)))-y_C/(2(x_C-1)))$ $= 0$ as desired.

□

The theorem of the Exeter point follows from the above by taking $E = A$.

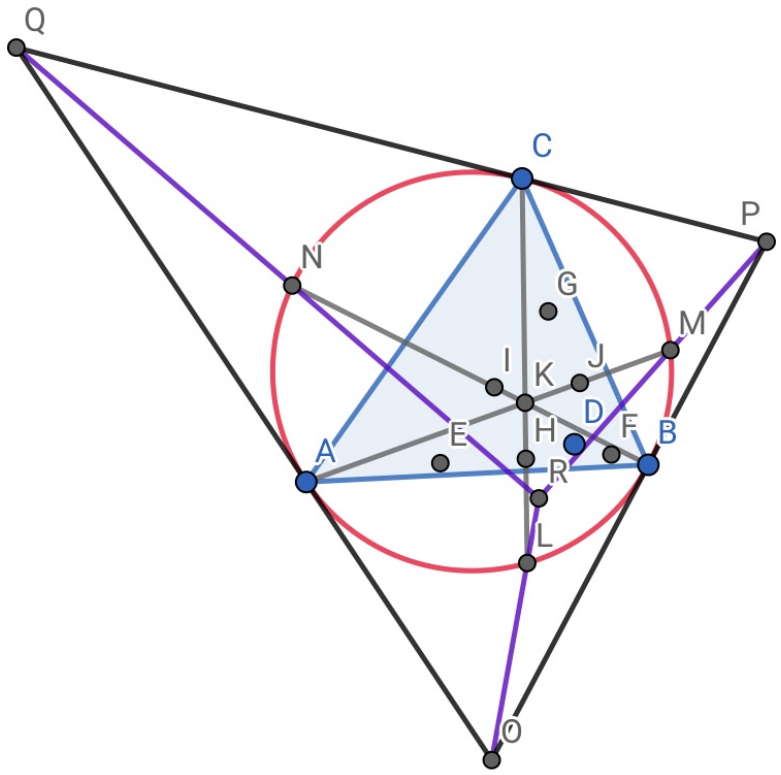

FIGURE 1. Illustration of theorem 2.1.

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