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Generalization of the Euler Line of a Quadrilateral

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1. Introduction

The Euler line of a triangle [1] is the line passing through the centroid, circumcenter, orthocenter, and nine point center. It is arguably the most iconic line discussed regarding triangles. There is an analogy of it for quadrilaterals as well [2]. Here, we will use the generalization of the centroid discussed in [3] and some motivation from [4] to derive a generalization for the theorem of the Euler line in a quadrilateral.

2. Results

Theorem 2.1. *Let $A, B, C,$ and D be the vertices of a given quadrilateral arranged counterclockwise.*
Let E be some point in the plane.
Let F be a point on line AE.
Let G be the point on line BE such that $BG : EG = AF : EF$.
Let H be the point on line CE such that $CH : EH = AF : EF$.
Let I be the point on line DE such that $DI : EI = AF : EF$.
Let J be the midpoint of F and G.
Let K be the midpoint of G and H.
Let L be the midpoint of H and I.
Let M be the midpoint of F and I.
Let N be the midpoint of F and H.
Let O be the midpoint of G and I.

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Let lines AK, BN, and CJ intersect at point α_1.
Let lines BL, CO, and DK intersect at point α_2.
Let lines AL, CM, and DN intersect at point α_3.
Let lines AO, BM, and DJ intersect at point α_4.
Let lines $\alpha_1\alpha_3$ and $\alpha_2\alpha_4$ intersect at point P.
Let β_1 be the circumcenter of A, B, and C.
Let β_2 be the circumcenter of B, C, and D.
Let β_3 be the circumcenter of A, C, and D.
Let β_4 be the circumcenter of A, B, and D.
Let lines $\beta_1\beta_3$ and $\beta_2\beta_4$ intersect at point Q.
Let γ_1 denote the nine point center of F, G, and H.
Let γ_2 denote the nine point center of G, H, and I.
Let γ_3 denote the nine point center of F, H, and I.
Let γ_4 denote the nine point center of F, G, and I.
Let lines $\gamma_1\gamma_3$ and $\gamma_2\gamma_4$ intersect at point R.
Let Q, P, and R are collinear in this order and $PQ : PR = 2AE : EF$.

Proof. Without loss of generality, we will use Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates (x_E, y_E).
F has coordinates $(x_F, \frac{x_F y_E}{x_E})$.
G has coordinates $(1 + \frac{x_F(x_E-1)}{x_E}, \frac{x_F y_E}{x_E})$.
H has coordinates $(x_C + \frac{x_F(x_E-x_C)}{x_E}, y_C + \frac{x_F(y_E-y_C)}{x_E})$.
I has coordinates $(x_D + \frac{x_F(x_E-x_D)}{x_E}, \frac{x_F y_E+y_D(x_E-x_F)}{x_E})$.
J has coordinates $(\frac{1}{2}(x_F + 1 + \frac{x_F(x_E-1)}{x_E}), \frac{x_F y_E}{x_E})$.
K has coordinates $(\frac{x_E(x_C+1)+x_F(2x_E-x_C-1)}{2x_E}, \frac{2x_F y_E+y_C(x_E-x_F)}{2x_E})$.
L has coordinates $(\frac{x_E(x_C+x_D)+x_F(2x_E-x_D-x_C)}{2x_E}, \frac{2x_F y_E+(x_E-x_F)(y_C+y_D)}{2x_E})$.
M has coordinates $(\frac{1}{2}(x_F + x_D + \frac{x_F(x_E-x_D)}{x_E}), \frac{2x_F y_E+y_D(x_E-x_F)}{2x_E})$.
N has coordinates $(\frac{1}{2}(x_F + x_C + \frac{x_F(x_E-x_C)}{x_E}), \frac{2x_F y_E+y_C(x_E-x_F)}{2x_E})$.
O has coordinates $(\frac{x_E(x_D+1)+x_F(2x_E-x_D-1)}{2x_E}, \frac{2x_F y_E+y_D(x_E-x_F)}{2x_E})$.
α_1 has coordinates $(\frac{-x_C x_E+x_C x_F-2x_E x_F-x_E+x_F}{x_F-3x_E}, \frac{-x_E y_C+x_F y_C-2x_F y_E}{x_F-3x_E})$.
α_2 has coordinates $(\frac{x_E-x_F+x_C x_E-x_C x_F+x_D x_E-x_D x_F+2x_E x_F}{3x_E-x_F}, \frac{x_E y_C+x_E y_D-x_F y_C-x_F y_D+2x_F y_E}{3x_E-x_F})$.
α_3 has coordinates $(\frac{x_C x_E-x_C x_F+x_D x_E-x_D x_F+2x_E x_F}{3x_E-x_F}, \frac{x_E y_C+x_E y_D-x_F y_C-x_F y_D+2x_F y_E}{3x_E-x_F})$.
α_4 has coordinates $(\frac{x_E-x_F+x_D x_E-x_D x_F+2x_E x_F}{3x_E-x_F}, \frac{x_E y_D-x_F y_D+2x_F y_E}{3x_E-x_F})$.
P has coordinates
$(-\frac{x_C(2x_E x_F-x_D x_F-x_C x_F+x_D x_E+x_C x_E)y_D-(x_D-1)((2x_E-x_D-x_C-1)x_F+(x_D+x_C+1)x_E)y_C}{(x_F-3x_E)(x_C y_D-(x_D-1)y_C)}$,
$-\frac{2x_F(x_C y_D-x_D y_C+y_C)y_E-x_C(x_F-x_E)y_D^2+(x_D-x_C)(x_F-x_E)y_C y_D+(x_D-1)(x_F-x_E)y_C^2}{(x_F-3x_E)(x_C y_D-(x_D-1)y_C)})$.
β_1 has coordinates $(\frac{1}{2}, \frac{x_C(x_C-1)+y_C^2}{2y_C})$.
β_2 has coordinates
$(\frac{x_C^2 y_D-x_D^2 y_C+y_C^2 y_D-y_C y_D^2+y_C-y_D}{2(y_C-y_D+x_C y_D-x_D y_C)}, -\frac{x_C^2 x_D-x_C^2-x_C x_D^2-x_C y_D^2+x_C+x_D^2+x_D y_C^2-x_D-y_C^2+y_D^2}{2(y_C-y_D+x_C y_D-x_D y_C)})$.
β_3 has coordinates $(\frac{x_C^2 y_D-x_D^2 y_C+y_C^2 y_D-y_C y_D^2}{2(x_C y_D-x_D y_C)}, \frac{-x_C^2 x_D+x_C x_D^2+x_C y_D^2-x_D y_C^2}{2(x_C y_D-x_D y_C)})$.

β_4 has coordinates $(\frac{1}{2}, \frac{x_D(x_D-1)+y_D^2}{2y_D})$.
Q has coordinates
$(\frac{x_C^2y_D-x_D^2y_C+y_C^2y_D-y_Cy_D^2+y_C}{2(y_C+x_Cy_D-x_Dy_C)}, \frac{-x_C^2x_D+x_C^2+x_Cx_D^2+x_Cy_D^2-x_C-x_Dy_C^2+y_C^2}{2(y_C+x_Cy_D-x_Dy_C)})$.
γ_1 has coordinates
$(\frac{x_E-x_F+2x_Cx_E-2x_Cx_F+4x_Ex_F}{4x_E}, \frac{x_Cx_E-x_Cx_F-x_C^2x_E+x_C^2x_F+x_Ey_C^2-x_Fy_C^2+4x_Fy_Cy_E}{4x_Ey_C})$.
γ_2 has coordinates
$(-((x_F-x_E)y_Cy_D^2-(x_Fy_C^2-x_Ey_C^2+4x_Cx_Ex_F-4x_Ex_F-2x_Cx_Dx_F+2x_Dx_F-x_C^2x_F+x_F+(x_C-1)(2x_D+x_C+1)x_E)y_D+(x_D-1)((4x_E-x_D-2x_C-1)x_F+(x_D+2x_C+1)x_E)y_C)/(4x_E((x_C-1)y_D-(x_D-1)y_C)), (4x_F((x_C-1)y_D-(x_D-1)y_C)y_E-(x_C-1)(x_F-x_E)y_D^2+2(x_D-x_C)(x_F-x_E)y_Cy_D+(x_D-1)(x_F-x_E)y_C^2+(x_C-1)(x_D-1)(x_D-x_C)x_F-(x_C-1)(x_D-1)(x_D-x_C)x_E)/(4x_E((x_C-1)y_D-(x_D-1)y_C)))$.
γ_3 has coordinates
$(-((x_F-x_E)y_Cy_D^2+((x_E-x_F)y_C^2+(-4x_Cx_E+2x_Cx_D+x_C^2)x_F+(-2x_Cx_D-x_C^2)x_E)y_D+((4x_Dx_E-x_D^2-2x_Cx_D)x_F+(x_D^2+2x_Cx_D)x_E)y_C)/(4x_E(x_Cy_D-x_Dy_C)), (4x_F(x_Cy_D-x_Dy_C)y_E-x_C(x_F-x_E)y_D^2+2(x_D-x_C)(x_F-x_E)y_Cy_D+x_D(x_F-x_E)y_C^2+x_Cx_D(x_D-x_C)x_F-x_Cx_D(x_D-x_C)x_E)/(4x_E(x_Cy_D-x_Dy_C)))$.
γ_4 has coordinates
$(\frac{x_E-x_F+2x_Dx_E-2x_Dx_F+4x_Ex_F}{4x_E}, \frac{x_Dx_E-x_Dx_F-x_D^2x_E+x_D^2x_F+x_Ey_D^2-x_Fy_D^2+4x_Fy_Dy_E}{4x_Ey_D})$.
R has coordinates
$(-((x_F-x_E)y_Cy_D^2-(x_Fy_C^2-x_Ey_C^2+4x_Cx_Ex_F-2x_Cx_Dx_F-x_C^2x_F+2x_Cx_Dx_E+x_C^2x_E)y_D+(x_D-1)((4x_E-x_D-2x_C-1)x_F+(x_D+2x_C+1)x_E)y_C)/(4x_E(x_Cy_D-(x_D-1)y_C)), (4x_F(x_Cy_D-x_Dy_C+y_C)y_E-x_C(x_F-x_E)y_D^2+2(x_D-x_C)(x_F-x_E)y_Cy_D+(x_D-1)(x_F-x_E)y_C^2+x_C(x_D-1)(x_D-x_C+1)x_F-x_C(x_D-1)(x_D-x_C+1)x_E)/(4x_E(x_Cy_D-(x_D-1)y_C)))$.
Line PQ is given by the equation
$y=((-(2x_F(x_Cy_D-x_Dy_C+y_C)y_E-x_C(x_F-x_E)y_D^2+(x_D-x_C)(x_F-x_E)y_Cy_D+(x_D-1)(x_F-x_E)y_C^2)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C))-(x_Cy_D^2-x_Dy_C^2+y_C^2+x_Cx_D^2-x_C^2x_D+x_C^2-x_C)/(2(x_Cy_D-x_Dy_C+y_C)))(x+(x_C(2x_Ex_F-x_Dx_F-x_Cx_F+x_Dx_E+x_Cx_E)y_D-(x_D-1)((2x_E-x_D-x_C-1)x_F+(x_D+x_C+1)x_E)y_C)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C))))/(-(-y_Cy_D^2+y_C^2y_D+x_C^2y_D-x_D^2y_C+y_C)/(2(x_Cy_D-x_Dy_C+y_C))-(x_C(2x_Ex_F-x_Dx_F-x_Cx_F+x_Dx_E+x_Cx_E)y_D-(x_D-1)((2x_E-x_D-x_C-1)x_F+(x_D+x_C+1)x_E)y_C)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C)))-(2x_F(x_Cy_D-x_Dy_C+y_C)y_E-x_C(x_F-x_E)y_D^2+(x_D-x_C)(x_F-x_E)y_Cy_D+(x_D-1)(x_F-x_E)y_C^2)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C))$.
We then verify that R lies on this line since we calculate that
$((-(2x_F(x_Cy_D-x_Dy_C+y_C)y_E-x_C(x_F-x_E)y_D^2+(x_D-x_C)(x_F-x_E)y_Cy_D+(x_D-1)(x_F-x_E)y_C^2)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C))-(x_Cy_D^2-x_Dy_C^2+y_C^2+x_Cx_D^2-x_C^2x_D+x_C^2-x_C)/(2(x_Cy_D-x_Dy_C+y_C)))((-((x_F-x_E)y_Cy_D^2-(x_Fy_C^2-x_Ey_C^2+4x_Cx_Ex_F-2x_Cx_Dx_F-x_C^2x_F+2x_Cx_Dx_E+x_C^2x_E)y_D+(x_D-1)((4x_E-x_D-2x_C-1)x_F+(x_D+2x_C+1)x_E)y_C)/(4x_E(x_Cy_D-(x_D-1)y_C)))+(x_C(2x_Ex_F-x_Dx_F-x_Cx_F+x_Dx_E+x_Cx_E)y_D-(x_D-1)((2x_E-x_D-x_C-1)x_F+(x_D+x_C+1)x_E)y_C)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C))))/(-(-y_Cy_D^2+y_C^2y_D+x_C^2y_D-x_D^2y_C+y_C)/(2(x_Cy_D-x_Dy_C+y_C))-(x_C(2x_Ex_F-x_Dx_F-x_Cx_F+x_Dx_E+x_Cx_E)y_D-(x_D-1)((2x_E-x_D-x_C-1)x_F+(x_D+x_C+1)x_E)y_C)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C)))-(2x_F(x_Cy_D-x_Dy_C+y_C)y_E-x_C(x_F-x_E)y_D^2+(x_D-x_C)(x_F-x_E)y_Cy_D+(x_D-1)(x_F-x_E)y_C^2)/((x_F-3x_E)(x_Cy_D-(x_D-1)y_C))-((4x_F(x_Cy_D-x_Dy_C+y_C)y_E-x_C(x_F-x_E)y_D^2+2(x_D-x_C)(x_F-x_E)y_Cy_D+(x_D-1)(x_F-x_E)y_C^2+x_C(x_D-1)(x_D-x_C+1)x_F-x_C(x_D-1)(x_D-x_C+1)x_E)/(4x_E(x_Cy_D-(x_D-1)y_C)))$

$= 0$ as desired.
We then calculate that $PQ : PR = |\frac{x_P - x_Q}{x_P - x_R}|$
$= |(-(-y_C y_D^2 + y_C^2 y_D + x_C^2 y_D - x_D^2 y_C + y_C)/(2(x_C y_D - x_D y_C + y_C)) - (x_C(2x_E x_F - x_D x_F - x_C x_F + x_D x_E + x_C x_E)y_D - (x_D - 1)((2x_E - x_D - x_C - 1)x_F + (x_D + x_C + 1)x_E)y_C)/((x_F - 3x_E)(x_C y_D - (x_D - 1)y_C)))/(((x_F - x_E)y_C y_D^2 - (x_F y_C^2 - x_E y_C^2 + 4x_C x_E x_F - 2x_C x_D x_F - x_C^2 x_F + 2x_C x_D x_E + x_C^2 x_E)y_D + (x_D - 1)((4x_E - x_D - 2x_C - 1)x_F + (x_D + 2x_C + 1)x_E)y_C)/(4x_E(x_C y_D - (x_D - 1)y_C)) - (x_C(2x_E x_F - x_D x_F - x_C x_F + x_D x_E + x_C x_E)y_D - (x_D - 1)((2x_E - x_D - x_C - 1)x_F + (x_D + x_C + 1)x_E)y_C)/((x_F - 3x_E)(x_C y_D - (x_D - 1)y_C)))|$
$= |\frac{2x_E}{x_F - x_E}| = 2AE : EF.$

□

The theorem of the Euler line of a quadrilateral follows as the special case of this by taking $F = A$.

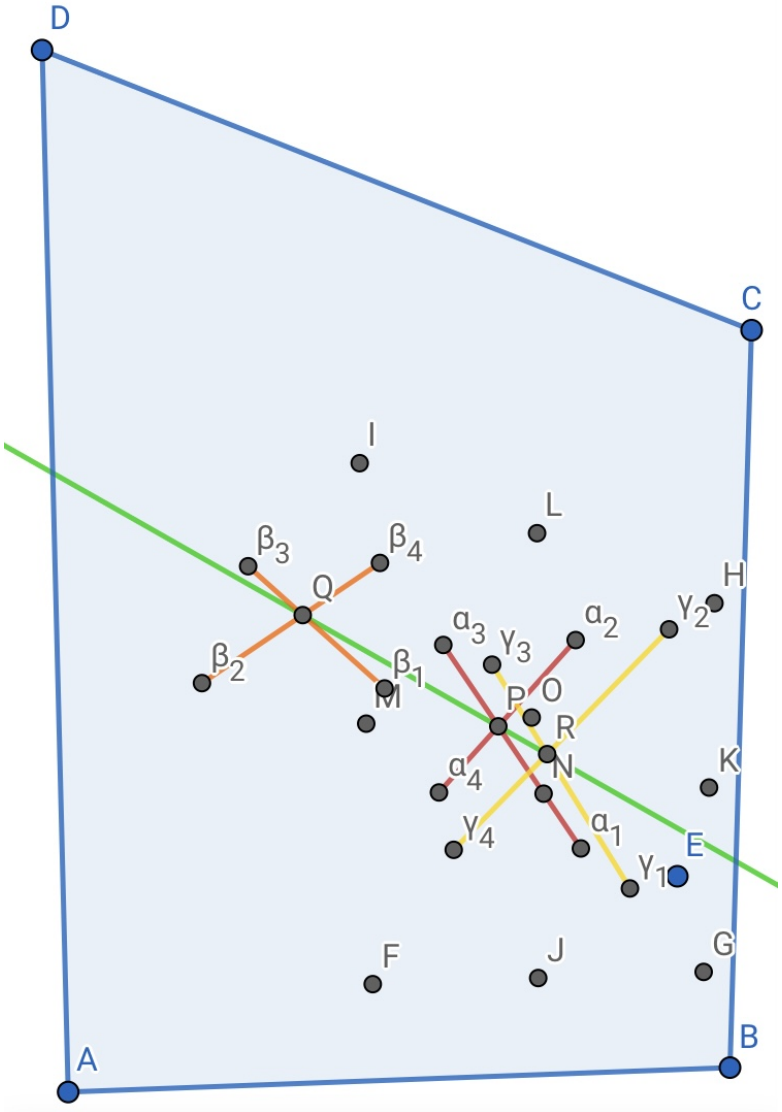

FIGURE 1. Illustration of theorem 2.1 where we take F to be the midpoint of A and E.

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