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Generalization of the Triangle Centroid and the Van Lamoen Circle

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1. Introduction

The Van Lamoen circle is a curious discovery only made 24 years ago. If you divide a triangle into six triangles by the three medians, then the six points which are the circumcenters of a vertex, the centroid, and the vertex midpoints are concyclic, which is the Van Lamoen circle.
Here we give two generalizations of this theorem and one for the triangle centroid.

2. Results

Theorem 2.1. *Let A, B, and C denote the vertices of a given triangle.*
Let D be any point in the plane.
Let E be a point on line AD.
Let F be a point on line BD such that $BF : DF = AE : DE$.
Let G be a point on line CD such that $CG : DG = AE : DE$.
Let H denote the midpoint of E and F.
Let I denote the midpoint of F and G.
Let J denote the midpoint of E and G.
Then
1. Lines AI, BJ, and CH are concurrent at point O,
2. $AO : IO = BO : JO = CO : HO = 2AD : DE$, and
3. The line formed between any two values of O (by varying the coordinates of E)

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passes through D.

Proof. The coordinates of A are $(0,0)$.
The coordinates of B are $(1,0)$.
The coordinates of C are (x_C, y_C).
The coordinates of D are (x_D, y_D).
The coordinates of E are $(x_E, \frac{x_E y_D}{x_D})$.
The coordinates of F are $(x_E - \frac{x_E}{x_D} + 1, \frac{x_E y_D}{x_D})$.
The coordinates of G are $(x_C + (\frac{x_E}{x_D})(x_D - x_C), y_C + (\frac{x_E}{x_D})(y_D - y_C))$.
The coordinates of H are $(\frac{1}{2}(2x_E - \frac{x_E}{x_D} + 1), \frac{x_E y_D}{x_D})$.
The coordinates of I are $(\frac{1}{2}(x_E - \frac{x_E}{x_D} + 1 + x_C + (\frac{x_E}{x_D})(x_D - x_C)), \frac{1}{2}(\frac{2x_E y_D + y_C(x_D - x_E)}{x_D}))$.
The coordinates of J are $(\frac{1}{2}(x_E + x_C + (\frac{x_E}{x_D})(x_D - x_C)), \frac{1}{2}(\frac{2x_E y_D + y_C(x_D - x_E)}{x_D}))$.
Line AI is given by the equation $y = x(\frac{2x_E y_D + y_C(x_D - x_E)}{x_E(2x_D - x_C - 1) + x_D(x_C + 1)})$.
Line BJ is given by the equation $y = (\frac{2x_E y_D + y_C(x_D - x_E)}{x_E(2x_D - x_C) + x_D(x_C - 2)})(x - 1)$.
Line CH is given by the equation $y = y_C + (\frac{y_C - \frac{x_E y_D}{x_D}}{x_C - (\frac{1}{2}(2x_E - \frac{x_E}{x_D} + 1))})(x - x_C)$.
We then verify that the point O given by $(\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E}, \frac{x_D y_C - x_E y_C + 2x_E y_D}{3x_D - x_E})$ lies on all three of these lines since we calculate
$(\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E})(\frac{2x_E y_D + y_C(x_D - x_E)}{x_E(2x_D - x_C - 1) + x_D(x_C + 1)}) - (\frac{x_D y_C - x_E y_C + 2x_E y_D}{3x_D - x_E}) = 0,$
$(\frac{2x_E y_D + y_C(x_D - x_E)}{x_E(2x_D - x_C) + x_D(x_C - 2)})(\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E} - 1) - (\frac{x_D y_C - x_E y_C + 2x_E y_D}{3x_D - x_E}) = 0,$
and
$y_C + (\frac{y_C - \frac{x_E y_D}{x_D}}{x_C - (\frac{1}{2}(2x_E - \frac{x_E}{x_D} + 1))})(\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E} - x_C) - (\frac{x_D y_C - x_E y_C + 2x_E y_D}{3x_D - x_E}) = 0.$
We next calculate that $AO : IO = BO : JO = CO : HO$
$= |\frac{\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E}}{\frac{1}{2}(x_E - \frac{x_E}{x_D} + 1 + x_C + (\frac{x_E}{x_D})(x_D - x_C)) - \frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E}}|$
$= |\frac{\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E} - 1}{\frac{1}{2}(x_E + x_C + (\frac{x_E}{x_D})(x_D - x_C)) - \frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E}}|$
$= |\frac{\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E} - x_C}{\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E} - \frac{1}{2}(2x_E - \frac{x_E}{x_D} + 1)}| = |\frac{2x_D}{x_D - x_E}|.$
Now, consider two of the values O and Q (corresponding to different values of x_E) given respectively by
$(\frac{x_C x_D - x_C x_E + 2x_D x_E + x_D - x_E}{3x_D - x_E}, \frac{x_D y_C - x_E y_C + 2x_E y_D}{3x_D - x_E})$
and
$(\frac{x_C x_D - x_C x_P + 2x_D x_P + x_D - x_P}{3x_D - x_P}, \frac{x_D y_C - x_P y_C + 2x_P y_D}{3x_D - x_P})$.
The line passing through both of these is given by the equation
$y = \frac{y_D(3x - x_C - 1) - y_C(x - x_D)}{3x_D - x_C - 1}$.
We then make the observation that the point (x_D, y_D) is a solution to this, so D and any two of these points are collinear.

□

The theorem of the triangle centroid is the special case of the previous theorem where we take $E = A$.

Theorem 2.2. *Let A, B, and C denote the vertices of a given triangle. Let D be the midpoint of A and B.*

Let E be the midpoint of A and C.
Let F be the midpoint of B and C.
Let G denote the centroid.
Let H be a point on line DG.
Let I be the point on line EG such that $EI : GI = DH : GH$.
Let J be the point on line FG such that $FJ : GJ = DH : GH$.
Let K be the circumcenter of A, G, and H.
Let L be the circumcenter of C, G, and J.
Let M be the circumcenter of A, G, and I.
Let N be the circumcenter of C, G, and I.
Let O be the circumcenter of B, G, and J.
Let P be the circumcenter of B, G, and H.
Then K, L, M, N, O, and P are concyclic, and all such resulting circles are coaxal with axis passing through the centroid.

Proof. A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates $(\frac{1}{2}, 0)$.
E has coordinates $(\frac{1}{2}x_C, \frac{1}{2}y_C)$.
F has coordinates $(\frac{1}{2}(x_C + 1), \frac{1}{2}y_C)$.
G has coordinates $(\frac{1}{3}(x_C + 1), \frac{1}{3}y_C)$.
H has coordinates $(x_H, \frac{y_C(2x_H-1)}{2x_C-1})$
I has coordinates $(\frac{x_H(2-x_C)+x_C^2-1}{2x_C-1}, \frac{y_C(x_C-x_H)}{2x_C-1})$.
J has coordinates $(\frac{(x_C+1)(x_C-x_H)}{2x_C-1}, \frac{y_C(x_C-x_H)}{2x_C-1})$.
K has coordinates
$(\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)}, \frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)})$.
L has coordinates
$(\frac{-2x_C^3-2x_C^2x_H+10x_C^2-4x_Cx_H-2x_Cy_C^2+2x_C-2x_Hy_C^2-2x_H+2y_C^2-1}{6(2x_C-1)},$
$\frac{x_C^3+x_C^2x_H-x_C^2+2x_Cx_H+x_Cy_C^2-2x_C+x_Hy_C^2+x_H+3y_C^2}{6y_C})$.
M has coordinates
$(\frac{-x_C^3+x_C^2x_H+4x_C^2-4x_Cx_H-x_Cy_C^2+2x_C+x_Hy_C^2+4x_H-3}{6(2x_C-1)},$
$\frac{x_C^4-x_C^3x_H-x_C^3+3x_C^2x_H+x_C^2y_C^2-3x_C^2-x_Cx_Hy_C^2+3x_Cy_C^2+x_C-x_Hy_C^2-4x_H-y_C^2+2}{6y_C(2x_C-1)})$.
N has coordinates
$(\frac{2x_C^3+2x_C^2x_H+2x_C^2-8x_Cx_H+2x_Cy_C^2+4x_C+2x_Hy_C^2+8x_H-2y_C^2-5}{6(2x_C-1)},$
$\frac{-x_C^3-x_C^2x_H+3x_C^2+4x_Cx_H-x_Cy_C^2-3x_C-x_Hy_C^2-4x_H+5y_C^2+2}{6y_C})$.
O has coordinates
$(\frac{x_C^3-x_C^2x_H+2x_C^2-2x_Cx_H+x_Cy_C^2+7x_C-x_Hy_C^2-x_H-3}{6(2x_C-1)},$
$\frac{-x_C^4+x_C^3x_H+2x_C^3-x_C^2y_C^2+x_Cx_Hy_C^2-3x_Cx_H+4x_Cy_C^2-x_C-2x_Hy_C^2-2x_H-y_C^2+2}{6y_C(2x_C-1)})$.
P has coordinates
$(\frac{4x_C^2x_H-2x_C^2-4x_Cx_H+11x_C+4x_Hy_C^2+x_H-2y_C^2-5}{6(2x_C-1)},$
$\frac{-4x_C^3x_H+4x_C^3+12x_C^2x_H-12x_C^2-4x_Cx_Hy_C^2-9x_Cx_H+4x_Cy_C^2+9x_C+8x_Hy_C^2+2x_H-5y_C^2-2}{6y_C(2x_C-1)})$.

Now consider the circle given by the equation
$(x - \frac{1}{36y_C^2}(x_C^5 + 3x_C^4x_H - 4x_C^4 - 6x_C^3x_H + 2x_C^3y_C^2 + 4x_C^3 + 6x_C^2x_Hy_C^2 - 6x_C^2y_C^2 + x_C^2 -$

$6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2+(y-\frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2 = (\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)}-\frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2+(\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)}-\frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2.$

We respectively verify that K, L, M, N, O, and P lie on this circle since we respectively calculate

$(\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)}-\frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2+(\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)}-\frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2-((\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)}-\frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2+(\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)}-\frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2)=0,$

$(\frac{-2x_C^3-2x_C^2x_H+10x_C^2-4x_Cx_H-2x_Cy_C^2+2x_C-2x_Hy_C^2-2x_H+2y_C^2-1}{6(2x_C-1)}-\frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2$
$+(\frac{x_C^3+x_C^2x_H-x_C^2+2x_Cx_H+x_Cy_C^2-2x_C+x_Hy_C^2+x_H+3y_C^2}{6y_C}-\frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2-((\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)}-\frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2+(\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)}-\frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2)=0,$

$(\frac{-x_C^3+x_C^2x_H+4x_C^2-4x_Cx_H-x_Cy_C^2+2x_C+x_Hy_C^2+4x_H-3}{6(2x_C-1)}-\frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2$
$+(\frac{x_C^4-x_C^3x_H-x_C^3+3x_C^2x_H+x_C^2y_C^2-3x_C^2-x_Cx_Hy_C^2+3x_Cy_C^2+x_C-x_Hy_C^2-4x_H-y_C^2+2}{6y_C(2x_C-1)}-\frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-$

$$3x_H - 4y_C^4 - 20y_C^2 + 2))^2 - ((\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)} - \frac{1}{36y_C^2}(x_C^5 + 3x_C^4x_H - 4x_C^4 - 6x_C^3x_H + 2x_C^3y_C^2 + 4x_C^3 + 6x_C^2x_Hy_C^2 - 6x_C^2y_C^2 + x_C^2 - 6x_Cx_Hy_C^2 + 3x_Cx_H + x_Cy_C^4 + 12x_Cy_C^2 - 4x_C + 3x_Hy_C^4 - 12x_Hy_C^2 - 6x_H - 2y_C^4 + 20y_C^2 + 4))^2 + (\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)} - \frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2) = 0,$$

$$(\frac{2x_C^3+2x_C^2x_H+2x_C^2-8x_Cx_H+2x_Cy_C^2+4x_C+2x_Hy_C^2+8x_H-2y_C^2-5}{6(2x_C-1)} - \frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2$$
$$+ (\frac{-x_C^3-x_C^2x_H+3x_C^2+4x_Cx_H-x_Cy_C^2-3x_C-x_Hy_C^2-4x_H+5y_C^2+2}{6y_C} - \frac{1}{36y_C(2x_C-1)}(2x_C^5 + 6x_C^4x_H - 8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H + 2x_Cy_C^4 + 36x_Cy_C^2 + 19x_C + 6x_Hy_C^4 + 12x_Hy_C^2 - 3x_H - 4y_C^4 - 20y_C^2 + 2))^2 - ((\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)} - \frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2+(\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)} - \frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2) = 0,$$

$$(\frac{x_C^3-x_C^2x_H+2x_C^2-2x_Cx_H+x_Cy_C^2+7x_C-x_Hy_C^2-x_H-3}{6(2x_C-1)} - \frac{1}{36y_C^2}(x_C^5 + 3x_C^4x_H - 4x_C^4 - 6x_C^3x_H + 2x_C^3y_C^2 + 4x_C^3 + 6x_C^2x_Hy_C^2 - 6x_C^2y_C^2 + x_C^2 - 6x_Cx_Hy_C^2 + 3x_Cx_H + x_Cy_C^4 + 12x_Cy_C^2 - 4x_C + 3x_Hy_C^4 - 12x_Hy_C^2 - 6x_H - 2y_C^4 + 20y_C^2 + 4))^2$$
$$+(\frac{-x_C^4+x_C^3x_H+2x_C^3-x_C^2y_C^2+x_Cx_Hy_C^2-3x_Cx_H+4x_Cy_C^2-x_C-2x_Hy_C^2-2x_H-y_C^2+2}{6y_C(2x_C-1)} - \frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H - 8x_C^4 - 12x_C^3x_H + 4x_C^3y_C^2 + 20x_C^3 + 12x_C^2x_Hy_C^2 + 36x_C^2x_H - 12x_C^2y_C^2 - 34x_C^2 - 12x_Cx_Hy_C^2 - 30x_Cx_H + 2x_Cy_C^4 + 36x_Cy_C^2 + 19x_C + 6x_Hy_C^4 + 12x_Hy_C^2 - 3x_H - 4y_C^4 - 20y_C^2 + 2))^2 - ((\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)} - \frac{1}{36y_C^2}(x_C^5 + 3x_C^4x_H - 4x_C^4 - 6x_C^3x_H + 2x_C^3y_C^2 + 4x_C^3 + 6x_C^2x_Hy_C^2 - 6x_C^2y_C^2 + x_C^2 - 6x_Cx_Hy_C^2 + 3x_Cx_H + x_Cy_C^4 + 12x_Cy_C^2 - 4x_C + 3x_Hy_C^4 - 12x_Hy_C^2 - 6x_H - 2y_C^4 + 20y_C^2 + 4))^2 + (\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)} - \frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2) = 0,$$

and

$$(\frac{4x_C^2x_H-2x_C^2-4x_Cx_H+11x_C+4x_Hy_C^2+x_H-2y_C^2-5}{6(2x_C-1)} - \frac{1}{36y_C^2}(x_C^5+3x_C^4x_H-4x_C^4-6x_C^3x_H+2x_C^3y_C^2+4x_C^3+6x_C^2x_Hy_C^2-6x_C^2y_C^2+x_C^2-6x_Cx_Hy_C^2+3x_Cx_H+x_Cy_C^4+12x_Cy_C^2-4x_C+3x_Hy_C^4-12x_Hy_C^2-6x_H-2y_C^4+20y_C^2+4))^2$$
$$+(\frac{-4x_C^3x_H+4x_C^3+12x_C^2x_H-12x_C^2-4x_Cx_Hy_C^2-9x_Cx_H+4x_Cy_C^2+9x_C+8x_Hy_C^2+2x_H-5y_C^2-2}{6y_C(2x_C-1)} - \frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H - 8x_C^4 - 12x_C^3x_H + 4x_C^3y_C^2 + 20x_C^3 + 12x_C^2x_Hy_C^2 + 36x_C^2x_H - 12x_C^2y_C^2 - 34x_C^2 - 12x_Cx_Hy_C^2 - 30x_Cx_H + 2x_Cy_C^4 + 36x_Cy_C^2 + 19x_C + 6x_Hy_C^4 + 12x_Hy_C^2 - 3x_H - 4y_C^4 - 20y_C^2 + 2))^2 - ((\frac{-4x_C^2x_H+2x_C^2+4x_Cx_H+x_C-4x_Hy_C^2-x_H+2y_C^2-1}{6(2x_C-1)} - \frac{1}{36y_C^2}(x_C^5 + 3x_C^4x_H - 4x_C^4 - 6x_C^3x_H + 2x_C^3y_C^2 + 4x_C^3 + 6x_C^2x_Hy_C^2 - 6x_C^2y_C^2 + x_C^2 - 6x_Cx_Hy_C^2 +$$

$3x_Cx_H + x_Cy_C^4 + 12x_Cy_C^2 - 4x_C + 3x_Hy_C^4 - 12x_Hy_C^2 - 6x_H - 2y_C^4 + 20y_C^2 + 4))^2 + (\frac{4x_C^3x_H+4x_Cx_Hy_C^2-3x_Cx_H+4x_Hy_C^2+x_H-3y_C^2}{6y_C(2x_C-1)} - \frac{1}{36y_C(2x_C-1)}(2x_C^5+6x_C^4x_H-8x_C^4-12x_C^3x_H+4x_C^3y_C^2+20x_C^3+12x_C^2x_Hy_C^2+36x_C^2x_H-12x_C^2y_C^2-34x_C^2-12x_Cx_Hy_C^2-30x_Cx_H+2x_Cy_C^4+36x_Cy_C^2+19x_C+6x_Hy_C^4+12x_Hy_C^2-3x_H-4y_C^4-20y_C^2+2))^2) = 0$ as desired. In the case that $x_C = \frac{1}{2}$ then we also take $x_H = \frac{1}{2}$.
Now, one can easily verify that all of these circles (formed by varying values of x_H) are coaxal, whose axis line passes through the centroid and is given by the equation $y = (y_C * ((2 * (y_C^2 * (y_C^2 + 2 * (x_C^2 - x_C + 1)) + (x_C - 1) * x_C * (x_C^2 - x_C + 5)) - 1) * x - y_C^4 - 2 * x_C^2 * y_C^2 - 2 * x_C * y_C^2 - x_C^4 - 2 * x_C^3 + 2 * x_C + 1))/((2 * x_C - 1) * (y_C^2 * (y_C^2 + 2 * (x_C - 2) * (x_C + 1)) + (x_C - 2) * (x_C + 1) * (x_C^2 - x_C + 1)))$. □

The theorem of the Van Lamoen circle is the special case of the previous theorem where we take $H = D$.

The next result extends this further.

Theorem 2.3. *Let A, B, and C denote the vertices of a given triangle.*
Let D be any point in the plane.
Let E be a point on line AD.
Let F be the point on line CD such that $CF : DF = AE : DE$.
Let G be the point on line BD such that $BG : DG = AE : DE$.
Let H be the midpoint of E and F.
Let I be the midpoint of E and G.
Let J be the midpoint of F and G.
Let K be the point described in theorem 2.1 (point of concurrency of lines AJ, BH, and CI).
Let L be the circumcenter of A, I, and K.
Let M be the circumcenter of A, H, and K.
Let N be the circumcenter of C, J, and K.
Let O be the circumcenter of C, H, and K.
Let P be the circumcenter of B, I, and K.
Let Q be the circumcenter of B, J, and K.
Then L, M, N, O, P, and Q lie on the same conic section.

Proof. A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $(x_E, \frac{x_Ey_D}{x_D})$.
F has coordinates $(x_C + (\frac{x_E}{x_D})(x_D - x_C), y_C + (\frac{x_E}{x_D})(y_D - y_C))$.
G has coordinates $(x_E - \frac{x_E}{x_D} + 1, \frac{x_Ey_D}{x_D})$.
H has coordinates $(\frac{1}{2}(x_E + x_C + (\frac{x_E}{x_D})(x_D - x_C)), \frac{1}{2}(\frac{2x_Ey_D+y_C(x_D-x_E)}{x_D}))$.
I has coordinates $(\frac{1}{2}(2x_E - \frac{x_E}{x_D} + 1), \frac{x_Ey_D}{x_D})$.
J has coordinates $(\frac{1}{2}(x_E - \frac{x_E}{x_D} + 1 + x_C + (\frac{x_E}{x_D})(x_D - x_C)), \frac{1}{2}(\frac{2x_Ey_D+y_C(x_D-x_E)}{x_D}))$.
K has coordinates $(\frac{x_Cx_D-x_Cx_E+2x_Dx_E+x_D-x_E}{3x_D-x_E}, \frac{x_Dy_C-x_Ey_C+2x_Ey_D}{3x_D-x_E})$.
L has coordinates
$x_L = (4x_C^2x_D^2x_Ey_D - 4x_C^2x_Dx_E^2y_D + 16x_Cx_D^2x_E^2y_D + 8x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 12x_D^3x_E^2y_C - 12x_D^3x_Ey_C - 3x_D^3y_C + 4x_D^2x_E^3y_C - 8x_D^2x_E^3y_D + 16x_D^2x_E^2y_C - 8x_D^2x_E^2y_D + 4x_D^2x_Ey_C^2y_D + 7x_D^2x_Ey_C - 2x_D^2x_Ey_D - 4x_Dx_E^3y_C + 8x_Dx_E^3y_D - 4x_Dx_E^2y_C^2y_D + 4x_Dx_E^2y_Cy_D^2 -$

$5x_Dx_E^2y_C + 4x_Dx_E^2y_D + 4x_E^3y_Cy_D^2 + x_E^3y_C - 8x_E^3y_D^3 - 2x_E^3y_D)/(24x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 24x_D^3x_Ey_C - 12x_D^3y_C + 8x_D^2x_E^2y_C + 16x_D^2x_Ey_C - 4x_Dx_E^2y_C)$
and
$y_L = (-4x_C^2x_D^3x_E - 2x_C^2x_D^3 + 4x_C^2x_D^2x_E^2 + 4x_C^2x_D^2x_E - 2x_C^2x_Dx_E^2 - 4x_Cx_D^3x_E^2 - 4x_Cx_D^3x_E - x_Cx_D^3 - 4x_Cx_D^2x_E^3 + x_Cx_D^2x_E + 4x_Cx_Dx_E^3 + 12x_Cx_Dx_E^2y_D^2 + x_Cx_Dx_E^2 - 4x_Cx_E^3y_D^2 - x_Cx_E^3 + 8x_D^3x_E^3 + 12x_D^3x_E^2 - 4x_D^3x_Ey_C^2 + 6x_D^3x_E - 2x_D^3y_C^2 + x_D^3 - 12x_D^2x_E^3 + 4x_D^2x_E^2y_C^2 - 16x_D^2x_E^2y_Cy_D - 12x_D^2x_E^2 + 4x_D^2x_Ey_C^2 - 8x_D^2x_Ey_Cy_D - 3x_D^2x_E + 8x_Dx_E^3y_D^2 + 6x_Dx_E^3 - 2x_Dx_E^2y_C^2 + 8x_Dx_E^2y_Cy_D + 4x_Dx_E^2y_D^2 + 3x_Dx_E^2 - 4x_E^3y_D^2 - x_E^3)/(24x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 24x_D^3x_Ey_C - 12x_D^3y_C + 8x_D^2x_E^2y_C + 16x_D^2x_Ey_C - 4x_Dx_E^2y_C)$.
M has coordinates
$x_M = (-x_C^2x_D^2 + 2x_C^2x_Dx_E - x_C^2x_E^2 - 4x_Cx_D^2x_E + 4x_Cx_D^2 + 4x_Cx_Dx_E^2 - 4x_Cx_Dx_E - 4x_D^2x_E^2 + 8x_D^2x_E - x_D^2y_C^2 + 2x_D^2 + 2x_Dx_Ey_C^2 - 4x_Dx_Ey_Cy_D - 2x_Dx_E - x_E^2y_C^2 + 4x_E^2y_Cy_D - 4x_E^2y_D^2)/(12x_D^2 - 4x_Dx_E)$
and
$y_M = (x_C^3x_D^3 - 3x_C^3x_D^2x_E + 3x_C^3x_Dx_E^2 - x_C^3x_E^3 + 6x_C^2x_D^3x_E - x_C^2x_D^3 - 12x_C^2x_D^2x_E^2 + x_C^2x_D^2x_E + 6x_C^2x_Dx_E^3 + x_C^2x_Dx_E^2 - x_C^2x_E^3 + 12x_Cx_D^3x_E^2 - 4x_Cx_D^3x_E + x_Cx_D^3y_C^2 - 2x_Cx_D^3 - 12x_Cx_D^2x_E^3 - 3x_Cx_D^2x_Ey_C^2 + 4x_Cx_D^2x_Ey_Cy_D + 4x_Cx_D^2x_E + 4x_Cx_Dx_E^3 + 3x_Cx_Dx_E^2y_C^2 - 8x_Cx_Dx_E^2y_Cy_D + 4x_Cx_Dx_E^2y_D^2 - 2x_Cx_Dx_E^2 - x_Cx_E^3y_C^2 + 4x_Cx_E^3y_Cy_D - 4x_Cx_E^3y_D^2 + 8x_D^3x_E^3 - 4x_D^3x_E^2 + 2x_D^3x_Ey_C^2 - 4x_D^3x_E + 3x_D^3y_C^2 - 4x_D^2x_E^3 - 4x_D^2x_E^2y_C^2 + 8x_D^2x_E^2y_Cy_D + 4x_D^2x_E^2 - 7x_D^2x_Ey_C^2 + 12x_D^2x_Ey_Cy_D + 2x_Dx_E^3y_C^2 - 8x_Dx_E^3y_Cy_D + 8x_Dx_E^3y_D^2 + 5x_Dx_E^2y_C^2 - 16x_Dx_E^2y_Cy_D + 12x_Dx_E^2y_D^2 - x_E^3y_C^2 + 4x_E^3y_Cy_D - 4x_E^3y_D^2)/(12x_D^3y_C - 16x_D^2x_Ey_C + 24x_D^2x_Ey_D + 4x_Dx_E^2y_C - 8x_Dx_E^2y_D)$.
N has coordinates
$x_N = (x_C^2x_D^3y_C + 7x_C^2x_D^2x_Ey_D - x_C^2x_Dx_E^2y_C - 4x_C^2x_Dx_E^2y_D + x_C^2x_E^3y_D - 8x_Cx_D^3x_Ey_C - 4x_Cx_D^3y_C + 8x_Cx_D^2x_E^2y_C + 4x_Cx_D^2x_E^2y_D + 8x_Cx_D^2x_Ey_C + 2x_Cx_D^2x_Ey_D - 4x_Cx_Dx_E^3y_D - 4x_Cx_Dx_E^2y_C - 4x_Cx_Dx_E^2y_D + 2x_Cx_E^3y_D - 8x_D^3x_E^2y_C - 8x_D^3x_Ey_C + x_D^3y_C^3 - 2x_D^3y_C + 4x_D^2x_E^3y_D + 8x_D^2x_E^2y_C + 4x_D^2x_E^2y_D - x_D^2x_Ey_C^2y_D + 4x_D^2x_Ey_C + x_D^2x_Ey_D - 4x_Dx_E^3y_D - x_Dx_E^2y_C^3 + 4x_Dx_E^2y_C^2y_D - 4x_Dx_E^2y_Cy_D^2 - 2x_Dx_E^2y_C - 2x_Dx_E^2y_D + x_E^3y_C^2y_D - 4x_E^3y_Cy_D^2 + 4x_E^3y_D^3 + x_E^3y_D)/(12x_Cx_D^2x_Ey_D - 4x_Cx_Dx_E^2y_D - 12x_D^3x_Ey_C - 6x_D^3y_C + 4x_D^2x_E^2y_C + 8x_D^2x_Ey_C - 2x_Dx_E^2y_C)$
and
$y_N = (-2x_C^3x_D^3 + 2x_C^3x_Dx_E^2 + 2x_C^2x_D^3x_E + x_C^2x_D^3 - 8x_C^2x_D^2x_E^2 - 5x_C^2x_D^2x_E - 2x_C^2x_Dx_E^3 + 3x_C^2x_Dx_E^2 + x_C^2x_E^3 + 8x_Cx_D^3x_E^2 + 8x_Cx_D^3x_E - 2x_Cx_D^3y_C^2 + 2x_Cx_D^3 + 8x_Cx_D^2x_E^3 + 16x_Cx_D^2x_Ey_Cy_D - 2x_Cx_D^2x_E - 8x_Cx_Dx_E^3 + 2x_Cx_Dx_E^2y_C^2 - 16x_Cx_Dx_E^2y_Cy_D + 16x_Cx_Dx_E^2y_D^2 - 2x_Cx_Dx_E^2 + 2x_Cx_E^3 - 8x_D^3x_E^3 - 12x_D^3x_E^2 - 14x_D^3x_Ey_C^2 - 6x_D^3x_E - 7x_D^3y_C^2 - x_D^3 + 12x_D^2x_E^3 + 8x_D^2x_E^2y_C^2 - 8x_D^2x_E^2y_Cy_D + 12x_D^2x_E^2 + 11x_D^2x_Ey_C^2 - 4x_D^2x_Ey_Cy_D + 3x_D^2x_E - 2x_Dx_E^3y_C^2 + 8x_Dx_E^3y_Cy_D - 8x_Dx_E^3y_D^2 - 6x_Dx_E^3 - 5x_Dx_E^2y_C^2 + 8x_Dx_E^2y_Cy_D - 4x_Dx_E^2y_D^2 - 3x_Dx_E^2 + x_E^3y_C^2 - 4x_E^3y_Cy_D + 4x_E^3y_D^2 + x_E^3)/(24x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 24x_D^3x_Ey_C - 12x_D^3y_C + 8x_D^2x_E^2y_C + 16x_D^2x_Ey_C - 4x_Dx_E^2y_C)$.
O has coordinates
$x_O = (x_C^2x_D^3y_C + 7x_C^2x_D^2x_Ey_D - x_C^2x_Dx_E^2y_C - 4x_C^2x_Dx_E^2y_D + x_C^2x_E^3y_D - 8x_Cx_D^3x_Ey_C + 2x_Cx_D^3y_C + 8x_Cx_D^2x_E^2y_C + 4x_Cx_D^2x_E^2y_D - 4x_Cx_D^2x_Ey_D - 4x_Cx_Dx_E^3y_D - 2x_Cx_Dx_E^2y_C + 4x_Cx_Dx_E^2y_D - 8x_D^3x_E^2y_C + 4x_D^3x_Ey_C + x_D^3y_C^3 + x_D^3y_C + 4x_D^2x_E^3y_D + 4x_D^2x_E^2y_C - 8x_D^2x_E^2y_D - x_D^2x_Ey_C^2y_D - 2x_D^2x_Ey_D - x_Dx_E^2y_C^3 + 4x_Dx_E^2y_C^2y_D - 4x_Dx_E^2y_Cy_D^2 - x_Dx_E^2y_C + 2x_Dx_E^2y_D + x_E^3y_C^2y_D - 4x_E^3y_Cy_D^2 + 4x_E^3y_D^3)/(12x_Cx_D^2x_Ey_D - 4x_Cx_Dx_E^2y_D - 12x_D^3x_Ey_C + 6x_D^3y_C + 4x_D^2x_E^2y_C + 4x_D^2x_Ey_C - 12x_D^2x_Ey_D - 2x_Dx_E^2y_C + 4x_Dx_E^2y_D)$
and
$y_O = (-2x_C^3x_D^3 + 2x_C^3x_Dx_E^2 + 2x_C^2x_D^3x_E + 5x_C^2x_D^3 - 8x_C^2x_D^2x_E^2 + 3x_C^2x_D^2x_E - 2x_C^2x_Dx_E^3 -$

$x_C^2x_Dx_E^2 + x_C^2x_E^3 + 8x_Cx_D^3x_E^2 - 12x_Cx_D^3x_E - 2x_Cx_D^3y_C^2 - 2x_Cx_D^3 + 8x_Cx_D^2x_E^3 + 16x_Cx_D^2x_Ey_Cy_D - 4x_Cx_Dx_E^3 + 2x_Cx_Dx_E^2y_C^2 - 16x_Cx_Dx_E^2y_Cy_D + 16x_Cx_Dx_E^2y_D^2 + 2x_Cx_Dx_E^2 - 8x_D^3x_E^3 + 4x_D^3x_E^2 - 14x_D^3x_Ey_C^2 + 4x_D^3x_E + 9x_D^3y_C^2 + 4x_D^2x_E^3 + 8x_D^2x_E^2y_C^2 - 8x_D^2x_E^2y_Cy_D - 4x_D^2x_E^2 + 3x_D^2x_Ey_C^2 - 12x_D^2x_Ey_Cy_D - 2x_Dx_E^3y_C^2 + 8x_Dx_E^3y_Cy_D - 8x_Dx_E^3y_D^2 - 5x_Dx_E^2y_C^2 + 16x_Dx_E^2y_Cy_D - 12x_Dx_E^2y_D^2 + x_E^3y_C^2 - 4x_E^3y_Cy_D + 4x_E^3y_D^2)/(24x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 24x_D^3x_Ey_C + 12x_D^3y_C + 8x_D^2x_E^2y_C + 8x_D^2x_Ey_C - 24x_D^2x_Ey_D - 4x_Dx_E^2y_C + 8x_Dx_E^2y_D)$.

P has coordinates

$x_P = (4x_C^2x_D^2x_Ey_D - 4x_C^2x_Dx_E^2y_D + 16x_Cx_D^2x_E^2y_D + 8x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 12x_D^3x_E^2y_C - 12x_D^3x_Ey_C + 9x_D^3y_C + 4x_D^2x_E^3y_C - 8x_D^2x_E^3y_D + 16x_D^2x_E^2y_C - 8x_D^2x_E^2y_D + 4x_D^2x_Ey_C^2y_D + 3x_D^2x_Ey_C - 14x_D^2x_Ey_D - 4x_Dx_E^3y_C + 8x_Dx_E^3y_D - 4x_Dx_E^2y_C^2y_D + 4x_Dx_E^2y_Cy_D^2 - 5x_Dx_E^2y_C + 8x_Dx_E^2y_D + 4x_E^3y_Cy_D^2 + x_E^3y_C - 8x_E^3y_D^3 - 2x_E^3y_D)/(24x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 24x_D^3x_Ey_C + 12x_D^3y_C + 8x_D^2x_E^2y_C + 8x_D^2x_Ey_C - 24x_D^2x_Ey_D - 4x_Dx_E^2y_C + 8x_Dx_E^2y_D)$

and

$y_P = (-4x_C^2x_D^3x_E + 2x_C^2x_D^3 + 4x_C^2x_D^2x_E^2 - 2x_C^2x_Dx_E^2 - 4x_Cx_D^3x_E^2 + 12x_Cx_D^3x_E - 5x_Cx_D^3 - 4x_Cx_D^2x_E^3 - 3x_Cx_D^2x_E + 4x_Cx_Dx_E^3 + 12x_Cx_Dx_E^2y_D^2 + x_Cx_Dx_E^2 - 4x_Cx_E^3y_D^2 - x_Cx_E^3 + 8x_D^3x_E^3 - 8x_D^3x_E^2 - 4x_D^3x_Ey_C^2 - 2x_D^3x_E + 2x_D^3y_C^2 + 2x_D^3 - 8x_D^2x_E^3 + 4x_D^2x_E^2y_C^2 - 16x_D^2x_E^2y_Cy_D + 8x_D^2x_E^2 + 8x_D^2x_Ey_Cy_D + 8x_Dx_E^3y_D^2 + 2x_Dx_E^3 - 2x_Dx_E^2y_C^2 + 8x_Dx_E^2y_Cy_D - 16x_Dx_E^2y_D^2 - 2x_Dx_E^2)/(24x_Cx_D^2x_Ey_D - 8x_Cx_Dx_E^2y_D - 24x_D^3x_Ey_C + 12x_D^3y_C + 8x_D^2x_E^2y_C + 8x_D^2x_Ey_C - 24x_D^2x_Ey_D - 4x_Dx_E^2y_C + 8x_Dx_E^2y_D)$.

Q has coordinates

$x_Q = (x_C^2x_D^2 - 2x_C^2x_Dx_E + x_C^2x_E^2 + 4x_Cx_D^2x_E + 2x_Cx_D^2 - 4x_Cx_Dx_E^2 - 4x_Cx_Dx_E + 2x_Cx_E^2 + 4x_D^2x_E^2 + 4x_D^2x_E + x_D^2y_C^2 + 7x_D^2 - 4x_Dx_E^2 - 2x_Dx_Ey_C^2 + 4x_Dx_Ey_Cy_D - 4x_Dx_E + x_E^2y_C^2 - 4x_E^2y_Cy_D + 4x_E^2y_D^2 + x_E^2)/(12x_D^2 - 4x_Dx_E)$

and

$y_Q = (-x_C^3x_D^3 + 3x_C^3x_D^2x_E - 3x_C^3x_Dx_E^2 + x_C^3x_E^3 - 6x_C^2x_D^3x_E + 2x_C^2x_D^3 + 12x_C^2x_D^2x_E^2 - 2x_C^2x_D^2x_E - 6x_C^2x_Dx_E^3 - 2x_C^2x_Dx_E^2 + 2x_C^2x_E^3 - 12x_Cx_D^3x_E^2 + 8x_Cx_D^3x_E - x_Cx_D^3y_C^2 + x_Cx_D^3 + 12x_Cx_D^2x_E^3 + 3x_Cx_D^2x_Ey_C^2 - 4x_Cx_D^2x_Ey_Cy_D - 5x_Cx_D^2x_E - 8x_Cx_Dx_E^3 - 3x_Cx_Dx_E^2y_C^2 + 8x_Cx_Dx_E^2y_Cy_D - 4x_Cx_Dx_E^2y_D^2 + 3x_Cx_Dx_E^2 + x_Cx_E^3y_C^2 - 4x_Cx_E^3y_Cy_D + 4x_Cx_E^3y_D^2 + x_Cx_E^3 - 8x_D^3x_E^3 + 8x_D^3x_E^2 - 2x_D^3x_Ey_C^2 + 2x_D^3x_E + 4x_D^3y_C^2 - 2x_D^3 + 8x_D^2x_E^3 + 4x_D^2x_E^2y_C^2 - 8x_D^2x_E^2y_Cy_D - 8x_D^2x_E^2 - 8x_D^2x_Ey_C^2 + 16x_D^2x_Ey_Cy_D - 2x_Dx_E^3y_C^2 + 8x_Dx_E^3y_Cy_D - 8x_Dx_E^3y_D^2 - 2x_Dx_E^3 + 4x_Dx_E^2y_C^2 - 16x_Dx_E^2y_Cy_D + 16x_Dx_E^2y_D^2 + 2x_Dx_E^2)/(12x_D^3y_C - 16x_D^2x_Ey_C + 24x_D^2x_Ey_D + 4x_Dx_E^2y_C - 8x_Dx_E^2y_D)$.

One can then make the tedious but finite calculation that

$$det\begin{bmatrix} x_L^2 & y_L^2 & x_Ly_L & x_L & y_L & 1 \\ x_M^2 & y_M^2 & x_My_M & x_M & y_M & 1 \\ x_N^2 & y_N^2 & x_Ny_N & x_N & y_N & 1 \\ x_O^2 & y_O^2 & x_Oy_O & x_O & y_O & 1 \\ x_P^2 & y_P^2 & x_Py_P & x_P & y_P & 1 \\ x_Q^2 & y_Q^2 & x_Qy_Q & x_Q & y_Q & 1 \end{bmatrix} = 0$$

as desired. □

REFERENCES

[1] Weisstein, Eric W. "van Lamoen Circle." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/vanLamoenCircle.html

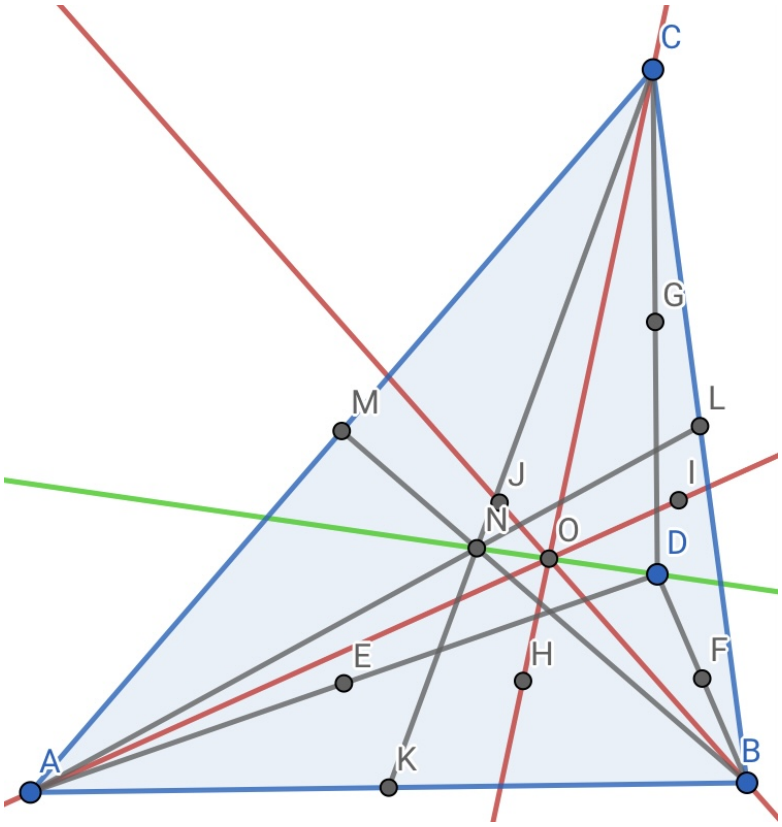

Figure 1. Illustration of theorem 2.1

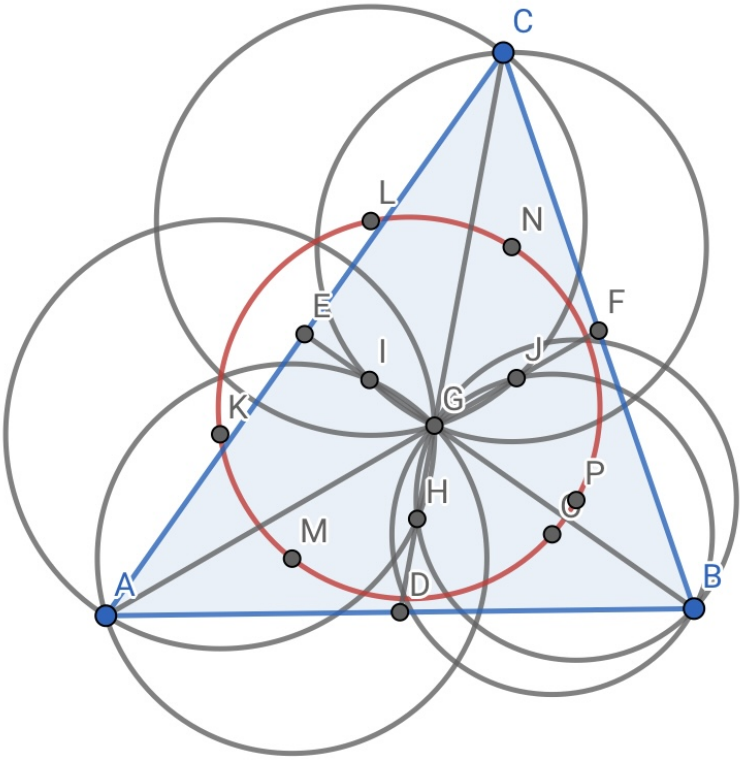

Figure 2. Illustration of theorem 2.2

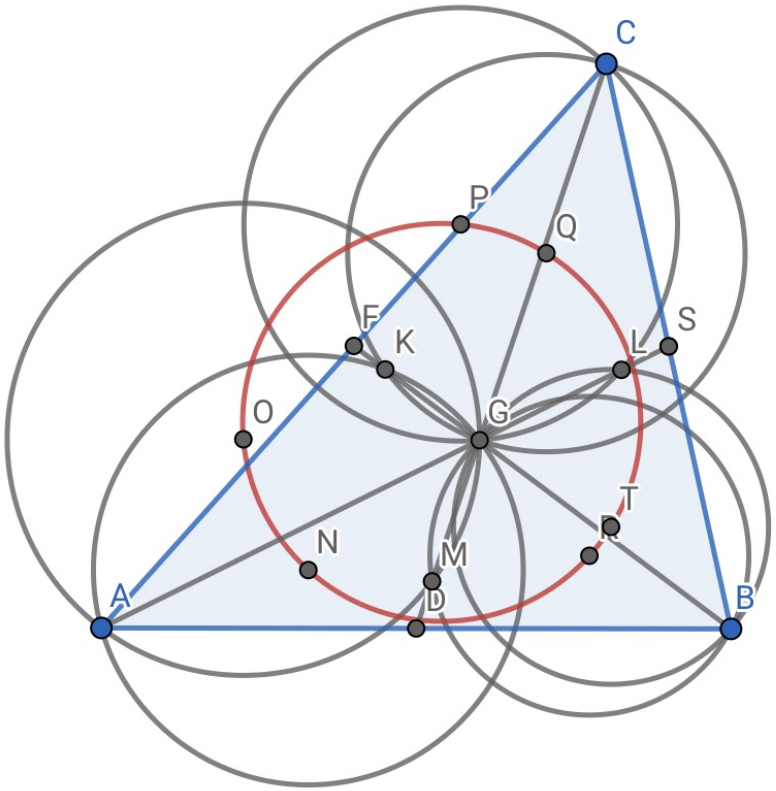

FIGURE 3. Further illustration of theorem 2.2 with relabeled points

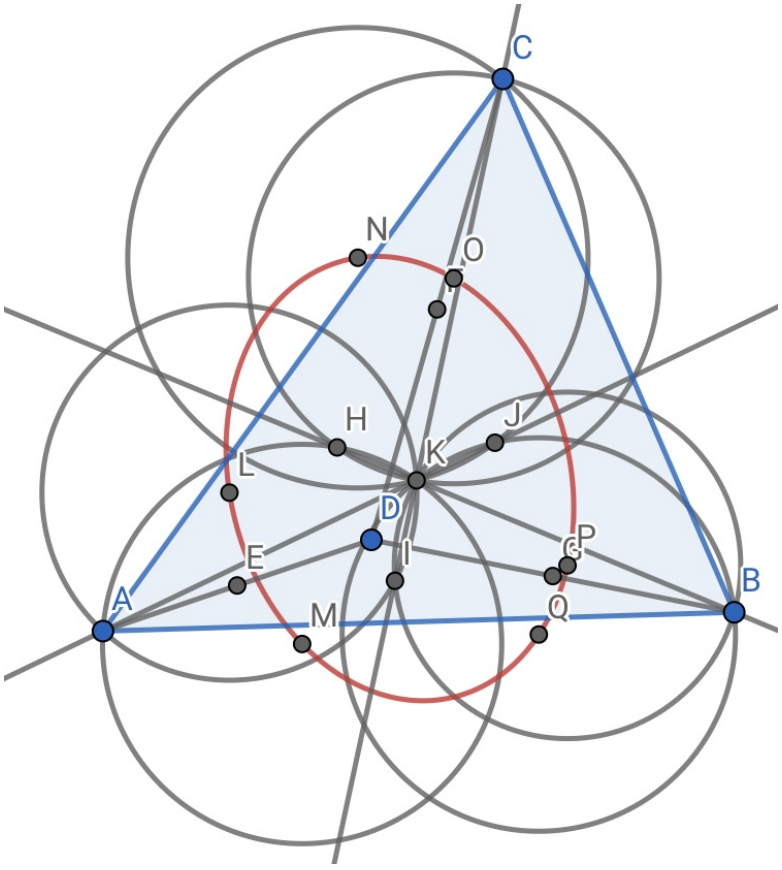

FIGURE 4. Illustration of theorem 2.3