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Some new Archimedean circles in an Arberlos

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Abstract. This article will refer to some new Archimedean 's circles in an arbelos.

Keywords. shoemaker's knife, arbelos, Archimedean circle.

1. Introduction

Given a segment AB with an interior point C. A shoemaker's knife (or arbelos) is the region obtained by cutting out from a semicircle (O) with diameter AB and the two smaller semicircles (O_1) and (O_2) with diameters AC and CB respectively. The common tangent at C of the smaller semicircles intersect the large semicircle at D, and let (O') is the Midway semicircle with diameter O_1O_2, and the points P, P_1 and P_2 are the midpoints of the semicircles $(O), (O_1)$ and (O_2) respectively, all on the same side AB. Let $AC = 2a, CB = 2b$. The following remarkable theorem is due to Archimedes.

Theorem 1.1. (*Archimedes,* [1]). *The two circles each tangent to CD, the large semicircle and one of the smaller semicircles have equal radius $\frac{ab}{a+b}$.*

Circles with radius $t := \frac{ab}{a+b}$ are called Archimedean. They are congruent to the Archimedean twin circles.

2. Preliminaries

Theorem 2.1. *Let A' and B' be the reflections of A and B through B and A respectively. Then two circles each tangent to the ray CD, the large semicircle (O) and one of the circles $(A'C)$ and $(B'C)$ are Archimedean.*

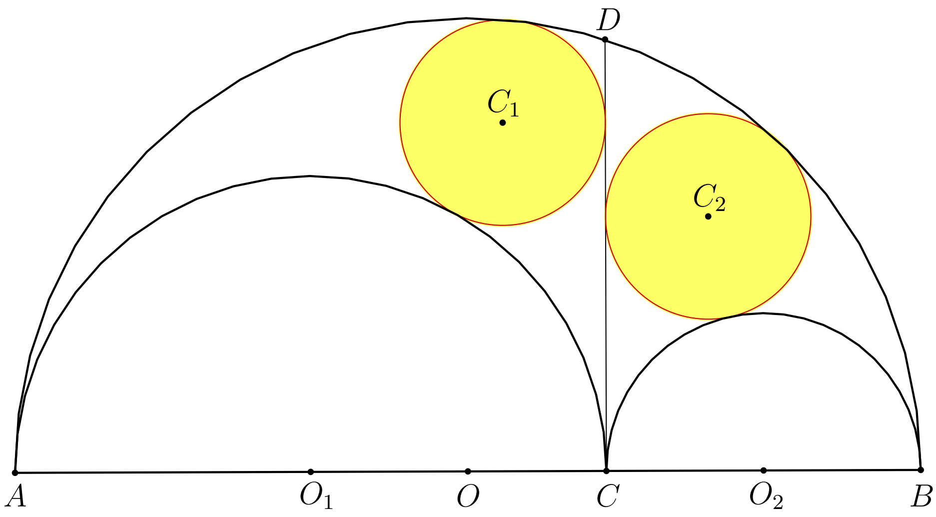

FIGURE 1

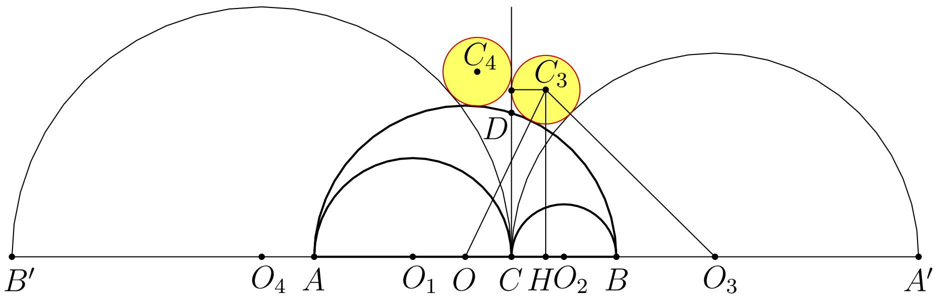

FIGURE 2

Proof. (see Figure 2). Let $(C_3), (C_4)$ are two circles each tangent to the ray CD, the large semicircle (O), with (C_3) tangent to $(A'C)$ and (C_4) tangent to $(B'C)$. Let r_3 is radius of circle (C_3); O_3 is center of $(A'C)$. Draw C_3O and C_3O_3, and drop perpendiculars from C_3 to line CD and point H to AB. We easily see that

$$C_3O = a + b + r_3,\ O_3C = a + 2b,\ C_3O_3 = a + 2b + r_3,$$
$$OH = a - b + r_3, \text{ and } O_3H = a + 2b - r_3.$$

From right triangles O_3C_3H and OC_3H we get that

$$C_3H^2 = (a + b + r_3)^2 - (a - b + r_3)^2 = (a + 2b + r_3)^2 - (a + 2b - r_3)^2,$$

which reduces to

$$2b(2a + 2r_3) = 2r_3(2a + 4b) \text{ and hence } r_3 = \frac{ab}{a + b} = t.$$

This means that (C_3) is Archimedean.
A similar argument show that (C_4) is Archimedean. □

Theorem 2.2. *The circle (PO_i), $(i = 1, 2)$ meets (O) at two points P and Q_i. Then circle tangent to large semicircle (O), the segments PO_i and O_iQ_i is Archimedean.*

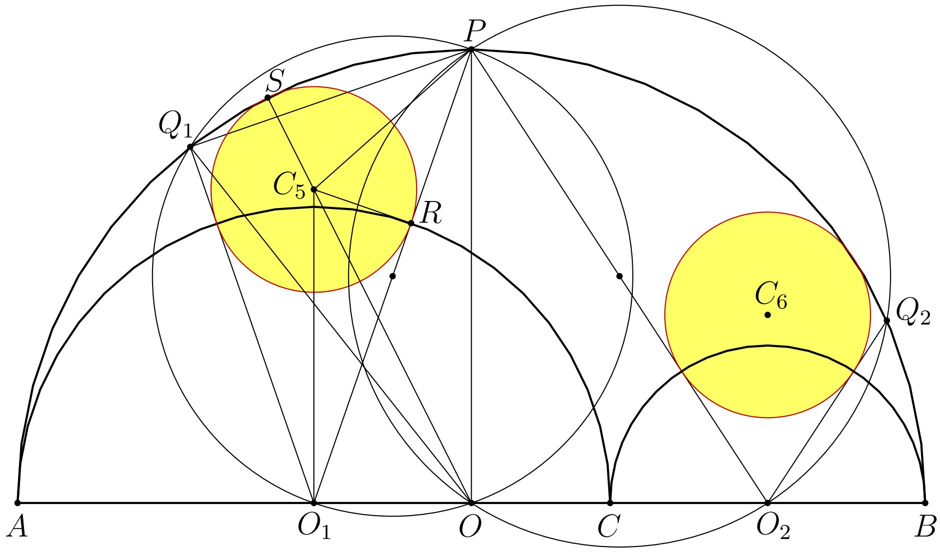

FIGURE 3

Proof. (see Figure 3). Let the circle (C_5) has radius r_5 tangent (O) at S, the segment PO_1 at R and the segment O_1Q_1.
We have $OO_1 = |AO - AO_1| = |(a + b) - a| = b$.

Since POO_1Q_1 is the cyclic quadrilateral, we have

$$\angle OO_1P = \angle OQ_1P = \angle OPQ_1 = \angle AO_1Q_1.$$

It follows that AO is the exterior angle bisector of $\angle PO_1Q_1$. Note that O_1C_5 is the interior angle bisector of $\angle PO_1Q_1$. Hence, O_1C_5 is perpendicular to AO. Thus, the right triangles O_1C_5R and PO_1O are similar,

$$\frac{O_1R}{C_5R} = \frac{PO}{O_1O} \Longrightarrow O_1R = C_5R\frac{PO}{O_1O} = r_5\frac{a+b}{b} = \frac{r_5(a+b)}{b}.$$

On the other hand, by (C_5) and (O) are internal tangent circles, C_5 belongs to the segment OS. It follows that $OC_5 = OS - C_5S = (a + b) - r_5$.

By the Pythagoras's theorem,

$$\begin{aligned} O_1R^2 &= O_1C_5^2 - C_5R^2 = O_1C_5^2 - r_5^2 = OC_5^2 - OO_1^2 - r_5^2 \\ &= [(a+b) - r_5]^2 - b^2 - r_5^2 = (a+b)^2 - 2r_5(a+b) - b^2. \end{aligned}$$

We deduce that,

$$\left(\frac{r_5(a+b)}{b}\right)^2 = (a+b)^2 - 2r_5(a+b) - b^2$$

$$\Longrightarrow r_5^2\frac{(a+b)^2}{b^2} + 2r_5(a+b) + [b^2 - (a+b)^2] = 0.$$

This implies $r_5 = \frac{ab}{a+b} = t$, and (C_5) is Archimedean circle; similarly for the circle (C_6) tangent to semicircle (O) and two segments PO_2 and O_2Q_2. $\square$

Remark. *The circle* (C_5) *(reps.* (C_6)*) is orthogonal to* (O_1) *(reps.* (O_2)*).*

Indeed, since $O_1R = r_5\frac{a+b}{b} = \frac{ab}{a+b}\frac{a+b}{b} = a$, we get R belongs to (O_1), and (C_5) is orthogonal to (O_1); similarly for (O_2) and (C_6).

Theorem 2.3. *Draw the circles* $O_1(O_2)$ *and* $O_2(O_1)$ *meet the semi-circle* (O) *at* T_1 *and* T_2*, respectively. The segments* T_1A, T_1O_1 *meet the semi-circle* (O_1) *at* C_7 *and* C_9*, respectively; the segments* T_2B, T_2O_2 *meet the semi-circle* (O_2) *at* C_8 *and* C_{10}*, respectively. Then*

(a) The cirles $C_7(C_9)$*,* $C_9(C_7)$*,* $C_8(C_{10})$*,* $C_{10}(C_8)$ *are Archimedean.*

(b) The circles, whose centers lie on the semi-circle (O)*, are tangent to* (O_1) *at* C_7*,* (O_2) *at* C_8 *are Archimedean, respectively.*

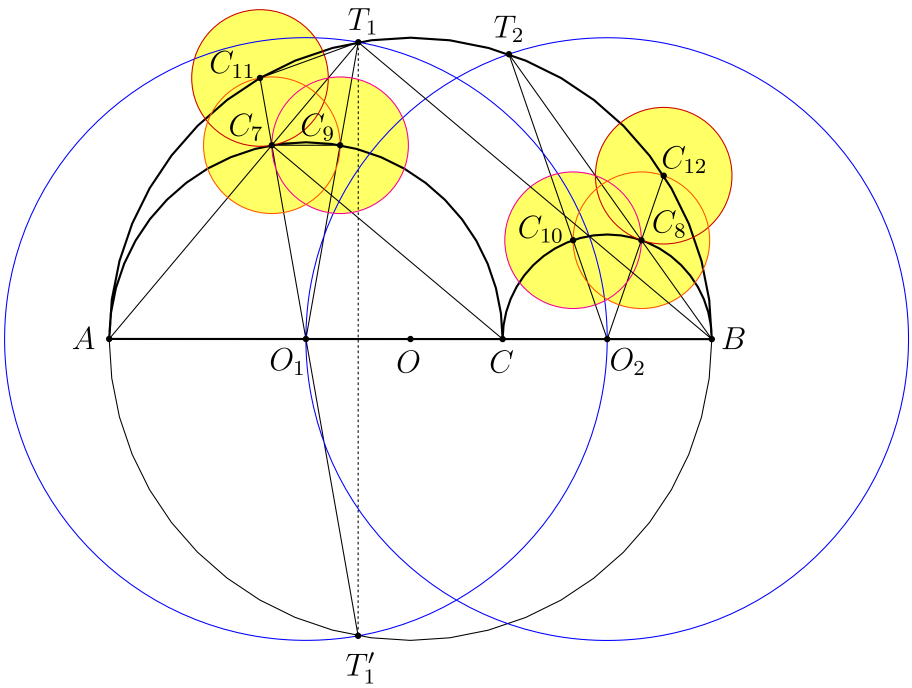

FIGURE 4

Proof. (see Figure 4). Since CC_7 and BT_1 are both perpendicular to AT_1, they are parallel.
By Thales's theorem,

$$\frac{AC_7}{AT_1} = \frac{AC}{AB} = \frac{2a}{2(a+b)} = \frac{a}{a+b} = \frac{O_1C_9}{O_1T_1}.$$

Again, by Thales's theorem, C_7C_9 is parallel to AO_1.

Since $T_1C_9 = T_1O_1 - O_1C_9 = (a+b) - a = b$, and the triangles $T_1C_7C_9$ and T_1AO_1 are similar,

$$\frac{C_7C_9}{AO_1} = \frac{T_1C_9}{T_1O_1} \Longrightarrow C_7C_9 = AO_1\frac{T_1C_9}{T_1O_1} = a\frac{b}{a+b} = \frac{ab}{a+b} = t.$$

This proves that the circle $C_7(C_9)$ and $C_9(C_7)$ are Archimedean circles; similarly for $C_8(C_{10})$ and $C_{10}(C_8)$. Part (a) is proved.

Let T_1' is the reflection point of T_1 through line AB; O_1C_7 and O_2C_8 meet semicircle (O) at C_{11} and C_{12}, respectively. We have $C_{11}(C_7)$ tangent to (O_1) at C_7, and $C_{12}(C_8)$ tangent to (O_2) at C_8.

By part (a), C_7C_9 is parallel to AB. It follows AB is the exterior angle bisector of $\angle T_1O_1C_{11}$. By symmetry, T_1' belongs to both the line $C_{11}O_1$ and the circle (O). Chassing angles,

$$\angle C_{11}C_7T_1 = \angle O_1C_7A = \angle O_1AC_7 = \angle C_9C_7T_1,$$

and

$$\begin{aligned}\angle T_1C_{11}C_7 &= \angle T_1C_{11}T_1' = \angle T_1AT_1' = 2\angle T_1AO_1\\ &= 2\angle T_1C_7C_9 = \angle C_{11}C_7C_9 = \angle T_1C_9C_7.\end{aligned}$$

This means that the triangles $T_1C_{11}C_7$ and $T_1C_9C_7$ are congruent. By part (a), $C_{11}C_7 = C_9C_7 = t$. This proves that $C_{11}(C_7)$ is Archimedean circle; similarly for $C_{12}(C_8)$. □

Theorem 2.4. *Draw the circles $A(O_1)$ and $B(O_2)$ meeting the semi-circle (O) at C_{13} and C_{14}, respectively. AC_{13} meets the semi-circle (O_1) at C_{15}, and BC_{14} meets the semi-circle (O_2) at C_{16}. Then the circles $C_{13}(C_{15})$, $C_{15}(C_{13})$, $C_{14}(C_{16})$ and $C_{16}(C_{14})$ are Archimedean.*

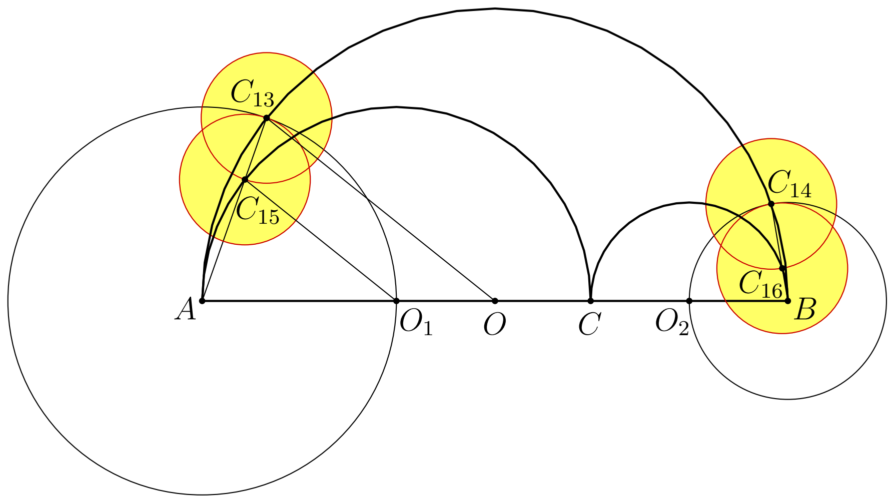

FIGURE 5

Proof. (see Figure 5). Connecting OC_{13} and O_1C_{15}. Since two circles (O) and (O_1) are tangent at A, the homothety $\mathcal{H}_A^{\frac{a}{a+b}}$ center A, ratio $\frac{a}{a+b}$ transforms (O) into (O_1). And by the homothety $\mathcal{H}_A^{\frac{a}{a+b}}$ then C_{13} goes into C_{15}. Hence

$$AC_{15} = \frac{a}{a+b}AC_{13} = \frac{a}{a+b}a = \frac{a^2}{a+b}.$$

It follows

$$C_{13}C_{15} = AC_{13} - AC_{15} = a - \frac{a^2}{a+b} = \frac{ab}{a+b} = t.$$

This thing proves that the circles $C_{13}(C_{15})$ and $C_{15}(C_{13})$ are Archimedean; similarly to the circles $C_{14}(C_{16})$ and $C_{16}(C_{14})$.

□

Theorem 2.5. *The line AD meets (O_1) at D_1, and BD meets (O_2) at D_2. Let C_{17} and C_{18} be points symmetric to O_1 and O_2 with respect to A and B. DC_{17} and DC_{18} meet the semi-circle (O) at R_1 and R_2, respectively. AR_1 meets (O_1) at S_1, and BR_2 meets (O_2) at S_2. D_1S_1 and D_2S_2 meet AB at C_{19} and C_{20}, respectively. Then the circles $C_{17}(C_{19})$, $C_{19}(C_{17})$, $C_{18}(C_{20})$ and $C_{20}(C_{18})$ are Archimedean.*

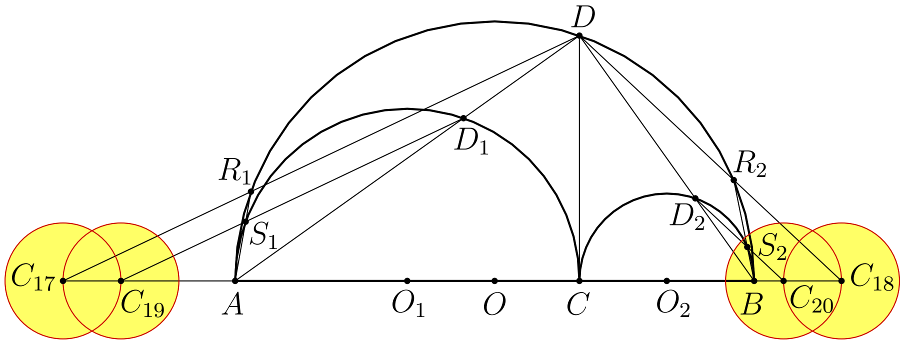

FIGURE 6

Proof. (see Figure 6). It is easy to have that under the homothety $\mathcal{H}_A^{\frac{a}{a+b}}$, the points D, R_1, C_{17} go into the points D_1, R_1, C_{19}, respectively. It follows

$$AC_{19} = \frac{a}{a+b}AC_{17} = \frac{a}{a+b}a = \frac{a^2}{a+b}.$$

Note that $AC_{17} = AO_1 = a$. Hence

$$C_{17}C_{19} = AC_{17} - AC_{19} = a - \frac{a^2}{a+b} = t.$$

This thing means that $C_{17}(C_{19})$ and $C_{19}(C_{17})$ are Archimedean; similarly to $C_{18}(C_{20})$ and $C_{20}(C_{18})$. □

Theorem 2.6. *Line AD meets (O_1) at D_1, and BD meets (O_2) at D_2. Let E_1 and E_2 be the points symmetric to C with respect to A and B, respectively. DE_1 and DE_2 meet the semi-circle (O) at F_1 and F_2. AF_1 meets the semi-circle (O_1) at G_1, and BF_2 meets the semi-circle (O_2) at G_2. D_1G_1 and D_2G_2 meet AB at H_1 and H_2, respectively. Then the circles (E_1H_1) and (E_2H_2) are Archimedean.*

Proof. (see Figure 7). It is easy to see that under the homothety $\mathcal{H}_A^{\frac{a}{a+b}}$, the points D, E_1, F_1 go into the points D_1, H_1, G_1, respectively. Note that $AE_1 = 2a$, it follows

$$AH_1 = \frac{a}{a+b}AE_1 = \frac{a}{a+b}2a = \frac{2a^2}{a+b}.$$

Hence

$$E_1H_1 = AE_1 - AH_1 = 2a - \frac{2a^2}{a+b} = \frac{2ab}{a+b} = 2t.$$

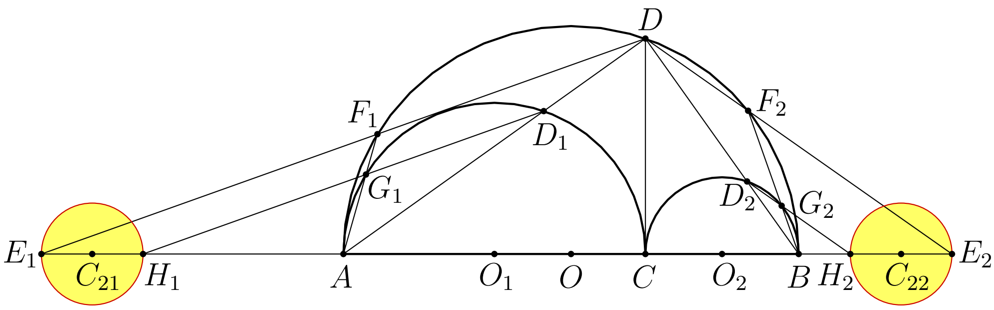

FIGURE 7

This thing means that the circle (E_1H_1) with center C_{21} is Archimedean. Similarly, the circle (E_2H_2) with center C_{22} is also Archimedean. □

Theorem 2.7. *Draw the circles $A(C)$ and $B(C)$ meeting the semi-circle (O) at J_1 and J_2. AB meets again $A(C)$ and $B(C)$ at E_1 and E_2, respectively. CJ_1 meets (O_1) at I_1, and CJ_2 meets (O_2) at I_2. E_1J_1 and E_2J_2 meet (O) at K_1 and K_2, respectively. AK_1 meets (O_1) at L_1, and BK_2 meets (O_2) at L_2. I_1L_1 meets K_1C and K_1E_1 at M_1 and N_1, respectively, and I_2L_2 meets K_2C and K_2E_2 at M_2 and N_2, respectively. Let $C_{23}, C_{24}, C_{25}, C_{26}$ be the midpoints of I_1A, I_2B, CL_1, CL_2. Then the circles $C_{23}(C_{25})$, $C_{25}(C_{23})$, $C_{24}(C_{26})$, $C_{26}(C_{24})$, (M_1L_1), (N_1L_1), (M_2L_2), (N_2L_2) are Archimedean.*

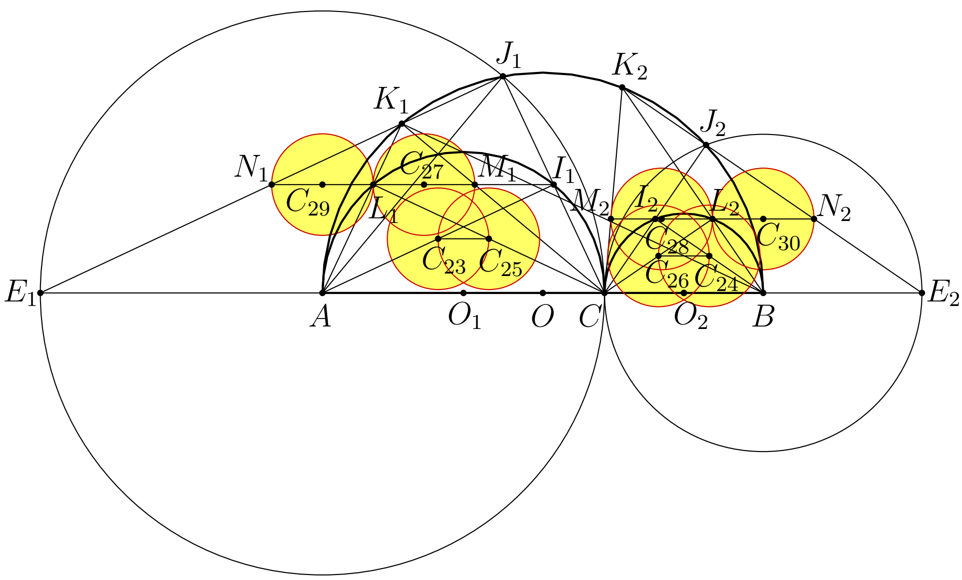

FIGURE 8

Proof. (see Figure 8). Let $C_{27}, C_{28}, C_{29}, C_{30}$ be the centers of circles (M_1L_1), (M_2L_2), (N_1L_1), (N_2L_2).

Connecting AJ_1 and BK_1. Since BK_1 and CL_1 are perpendicular to AK_1, they are parallel; and since AI_1 and E_1J_1 are perpendicular to CJ_1, they are parallel. Note that the quadrilateral ABJ_1K_1 is concyclic and triangle AJ_1E_1 is isosceles at A. Hence

$$\angle ACL_1 = \angle ABK_1 = \angle AJ_1K_1 = \angle AJ_1E_1 = \angle AE_1J_1 = \angle CAI_1.$$

This thing proves that ACI_1L_1 is an isosceles trapezoid and I_1L_1 is parallel to AC.

Since CL_1 is parallel to BK_1, applying the Thales's theorem, we have

$$\frac{AL_1}{AK_1} = \frac{AC}{AB} = \frac{a}{a+b} \Longrightarrow \frac{K_1L_1}{K_1A} = \frac{b}{a+b}.$$

Since M_1N_1 is parallel to AC, applying the Thales's theorem, we have

$$\frac{M_1L_1}{AC} = \frac{L_1N_1}{AE_1} = \frac{K_1L_1}{K_1A} = \frac{b}{a+b}.$$

Since $AC = AE_1 = 2a$, it follows $M_1L_1 = N_1L_1 = \frac{b}{a+b}2a = 2t$. This thing means that the circles (M_1L_1) and (N_1L_1) are Archimedean; similarly, the circles (M_2L_2) and (N_2L_2) are also Archimedean.

On the other hand, we have

$$I_1L_1 = N_1I_1 - N_1L_1 = AC - 2t = 2a - \frac{2ab}{a+b} = \frac{2a^2}{a+b}.$$

Hence

$$C_{23}C_{25} = \frac{AC - I_1L_1}{2} = \frac{2a - \frac{2a^2}{a+b}}{2} = \frac{ab}{a+b} = t.$$

This thing proves that $C_{23}(C_{25})$ and $C_{25}(C_{23})$ are Archimedean; similarly $C_{24}(C_{26})$ and $C_{26}(C_{24})$ are also Archimedean. □

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