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A Note on the Area of Triangles

Emmanuel A. J. García[a] [2]
[a] CIDIC-UTE,
C. Camila Henríquez Ureña 20, Santo Domingo, Dominican Republic
e-mail: emmanuelgeogarcia@gmail.com

Abstract. We show that the sum of the areas of two triangles obtained by reflection equals the area of the reference triangle.

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1. Introduction

On the web we can find several interesting problems associated with areas of triangles [1, 2, 3]. Theorem 1.1 is a generalization conjectured by Van Khea of a problem proposed by the author [4]. In this note we give a proof of Van Khea's generalization.

We will be using standard notation: $|BC| = a$, $|AC| = b$, $|AB| = c$; $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. If X, Y and Z are the vertices of a triangle, we denote its area $[XYZ]$.

Theorem 1.1 (Van Khea). *Let ABC be a triangle and P any point on the plane of ABC. Let X, Y and Z be arbitrary points on sides BC, AC and AB, respectively. Let D be the reflection of P around X. Similarly, define E and F. Denote U, V and W the midpoints of sides BC, AC and AB, respectively. Let D' be the reflection of D around U. Similarly, define E' and F'. Then*

$$[DEF] + [D'E'F'] = [ABC].$$

See figure 1 for an example of the situation described in Theorem 1.1.

[2]Corresponding author

Proof. Denote $AZ = g$, $BZ = h$, $BX = j$, $CX = k$, $CY = l$ and $AY = m$. The area of triangle XYZ can be expressed as follows

$$[XYZ] = \frac{1}{2}bc\sin\alpha - \frac{1}{2}gm\sin\alpha - \frac{1}{2}hj\sin\beta - \frac{1}{2}kl\sin\gamma.$$

Since triangles XYZ and DEF are homothetic with scale factor 2, it follows that

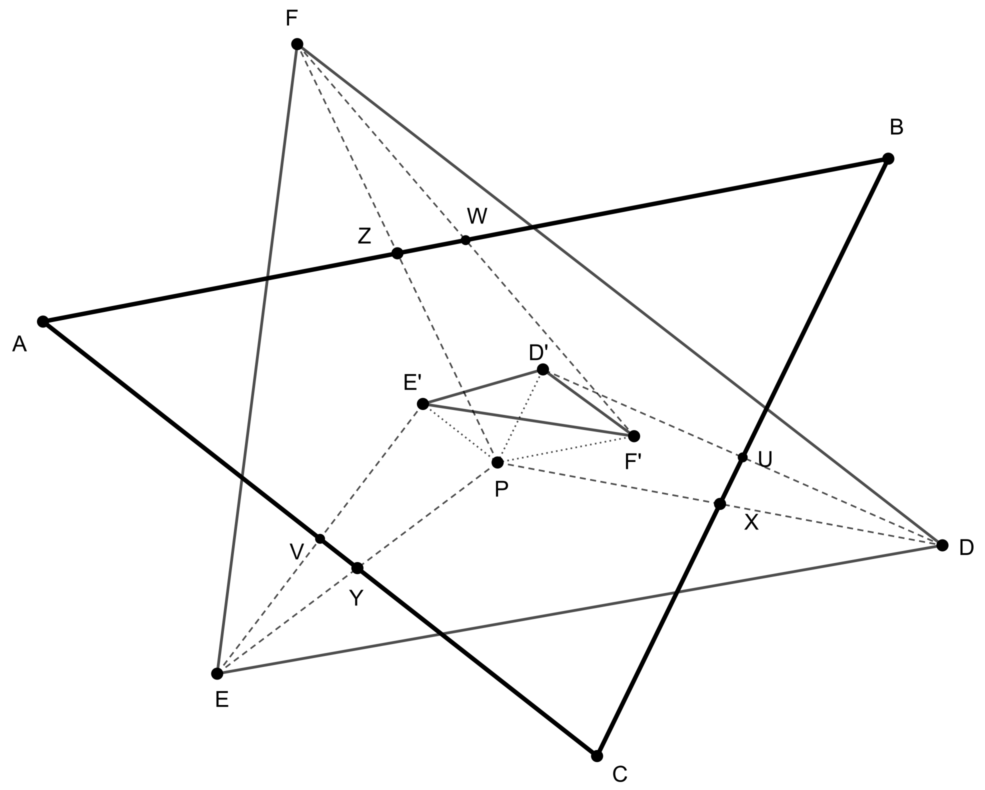

Figure 1. $[DEF] + [D'E'F'] = [ABC]$.

$[DEF] = 4[XYZ]$. Thus, we have

$$[DEF] = 2bc\sin\alpha - 2gm\sin\alpha - 2hj\sin\beta - 2kl\sin\gamma. \tag{1}$$

Dividing both sides of (1) by $[ABC]$ we have

$$\frac{[DEF]}{[ABC]} = 4\left[1 - \left(\frac{gm}{bc} + \frac{hj}{ca} + \frac{kl}{ab}\right)\right]. \tag{2}$$

Segments UX and PD' are homothetic with center at D and scale factor 2. It follows that

$$PD' = 2(j - \frac{1}{2}a) = 2j - a.$$

Similarly, we get $PE' = b - 2l$ and $PF' = c - 2g$. Since UX and PD' are homothetic segments, then UX and PD' are parallel and so are VY and PE'. Hence $\angle D'PE' = \gamma$. Similarly, $\angle D'PF' = \beta$. So the area of triangle $D'E'F'$ is given by the expression

$$\begin{aligned}[D'E'F'] &= [D'PE'] + [D'PF'] - [E'F'P] \\ &= \frac{(2j-a)(b-2l)\sin\gamma}{2} + \frac{(2j-a)(c-2g)\sin\beta}{2} - \frac{(b-2l)(c-2g)\sin(\beta+\gamma)}{2}.\end{aligned}$$

Taking into account that $\sin(\beta+\gamma) = \sin(\pi-\alpha) = \sin\alpha$ and dividing by $[ABC]$ we obtain

$$\frac{[D'E'F']}{[ABC]} = \frac{(2j-a)(b-2l)}{ab} + \frac{(2j-a)(c-2g)}{ca} - \frac{(b-2l)(c-2g)}{bc}. \tag{3}$$

Adding equations (2) and (3), expanding and factorizing,

$$\frac{[DEF]}{[ABC]} + \frac{[D'E'F']}{[ABC]} =$$
$$\frac{ca(4l+b) - 4ga(m+l-b) - 4j(b(g+h-c)+lc) - 4klc}{abc}.$$

But $b = m+l$, $c = g+h$ and $a = j+k$, so

$$\begin{aligned}\frac{[DEF]}{[ABC]} + \frac{[D'E'F']}{[ABC]} &= \frac{ca(4l+b) - 4jlc - 4klc}{abc}\\ &= \frac{ca(4l+b) - 4cl(j+k)}{abc}\\ &= \frac{ca(4l+b) - 4cla}{abc}\\ &= 1.\end{aligned}$$

Therefore,

$$[DEF] + [D'E'F'] = [ABC].$$

□

Remark. The point P may cross the side lines of the triangle ABC in points either interior or exterior to the sides. The reasoning in cases other than that considered above requires only minor adjustments.

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