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Trigonometric Identities and Urquhart's Theorem

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Abstract. The purpose of this note is to represent a general analytic geometry problem implying Urquhart's Theorem. The proofs are elementary and rely on some calculations and trigonometric identities.

Keywords. Urquhart's Theorem; Trigonometric Identities.

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1. Introduction

We start by stating this simple problem: In the Euclidean plane $(O, \overrightarrow{i}, \overrightarrow{j})$, take two concentric circles (A) and (B) of radius a and b, respectively, centered at O. Let $A \in (A)$ with $A(a\cos(\alpha); a\sin(\alpha))$ and $B \in (B)$ with $B(b\cos(\beta); b\sin(\beta))$ for some reals α and β in $[-\pi, \pi]$. For $C(r, 0)$, $r > 0$ the perpendicular bisectors of $[AC]$ and $[BC]$ intersect (OA) at A' and (OB) at B', respectively. Assuming that no opposite sides of the quadrilateral $OA'CB'$ are parallel, let $I = (A'C) \cap (OB')$ and $J = (B'C) \cap (OA')$. We fix all parameters except the radius b considered as a variable $b \geq 0$ and want to find b for which

$$|IO \pm IC| = |JO \pm JC|. \tag{1}$$

In this context we state the known Urquhart's theorem (see also [1]) as follows:

Theorem 1.1 ([3, 4])**.** *For $\alpha\beta \leq 0$, $a \geq r$ and $b = a$ we have $IO + IC = JO + JC$ (see Figure 1).*

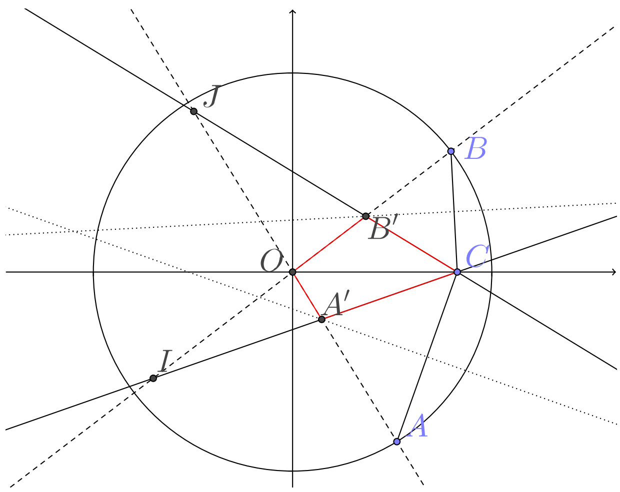


FIGURE 1. Urquhart's theorem

2. MAIN RESULTS

Considering the lines (OA) and (OB) of equations $y = \tan(\alpha)x$ and $y = \tan(\beta)x$, respectively, by straight calculations we have:
$A'\left(\frac{\cos(\alpha)(a^2-r^2)}{2a-2r\cos(\alpha)}, \frac{\sin(\alpha)(a^2-r^2)}{2a-2r\cos(\alpha)}\right)$ and thus $B'\left(\frac{\cos(\beta)(b^2-r^2)}{2b-2r\cos(\beta)}, \frac{\sin(\beta)(b^2-r^2)}{2b-2r\cos(\beta)}\right)$.
We also verify that the line (CA') is:

$$y = \left(\frac{\sin(\alpha)(a^2-r^2)}{(r^2+a^2)\cos(\alpha)-2ra}\right)x - \frac{r\sin(\alpha)(a^2-r^2)}{(r^2+a^2)\cos(\alpha)-2ar}.$$

The points I and J are:

$$I\begin{cases} x_I = \dfrac{r\cos(\beta)\sin(\alpha)(a^2-r^2)}{\cos(\beta)\sin(\alpha)(a^2-r^2)-\sin(\beta)((r^2+a^2)\cos(\alpha)-2ar)}, \\ \\ y_I = \dfrac{r\sin(\beta)\sin(\alpha)(a^2-r^2)}{\cos(\beta)\sin(\alpha)(a^2-r^2)-\sin(\beta)((r^2+a^2)\cos(\alpha)-2ar)}, \end{cases}$$

and by switching $\alpha \leftrightarrow \beta$, $a \leftrightarrow b$:

$$J\begin{cases} x_J = \dfrac{r\cos(\alpha)\sin(\beta)(b^2-r^2)}{\cos(\alpha)\sin(\beta)(b^2-r^2)-\sin(\alpha)((r^2+b^2)\cos(\beta)-2br)}, \\ \\ y_J = \dfrac{r\sin(\alpha)\sin(\beta)(b^2-r^2)}{\cos(\alpha)\sin(\beta)(b^2-r^2)-\sin(\alpha)((r^2+b^2)\cos(\beta)-2br)}. \end{cases}$$

Hereafter we asume that $\sin(\alpha)\sin(\beta) \neq 0$ since otherwise the configuration is trivial. The next lemma collects some identities that can be obtained at `www.dcode.fr/math-expression-factor`:

Lemma 2.1. *Let α, β, r and a be reals numbers, set $s = \frac{\alpha+\beta}{2}$ and $d = \frac{\alpha-\beta}{2}$:*

$$(\sin(\alpha)(a^2-r^2))^2 + ((r^2+a^2)\cos(\alpha)-2ar)^2 = (a^2+r^2-2ar\cos(\alpha))^2. \tag{2}$$

$$\begin{aligned} &r\sin(\alpha)(a^2-r^2) + r\sin(\beta)(a^2+r^2-2ar\cos(\alpha)) \\ &= 2r(a\sin(s)+r\sin(d))(a\cos(d)-r\cos(s)). \end{aligned} \tag{3}$$

$$r\sin(\alpha)(a^2-r^2)-r\sin(\beta)(a^2+r^2-2ar\cos(\alpha)) = 2r(r\sin(s)+a\sin(d))(a\cos(s)-r\cos(d)). \tag{4}$$

$$\cos(\beta)\sin(\alpha)(a^2-r^2)-\sin(\beta)((r^2+a^2)\cos(\alpha)-2ar) = 2(a\sin(d)+r\sin(s))(a\cos(d)-r\cos(s)). \tag{5}$$

Lemma 2.2. *With the given definitions we have:*

- $OA' = \dfrac{|a^2-r^2|}{|2a-2r\cos(\alpha)|}$, $OB' = \dfrac{|b^2-r^2|}{|2b-2r\cos(\beta)|}$,
- $CA' = \dfrac{a^2+r^2-2ar\cos(\alpha)}{|2a-2r\cos(\alpha)|}$ *and* $CB' = \dfrac{b^2+r^2-2br\cos(\beta)}{|2b-2r\cos(\beta)|}$,
- $OI = \dfrac{|r\sin(\alpha)(a^2-r^2)|}{|\cos(\beta)\sin(\alpha)(a^2-r^2)-\sin(\beta)((r^2+a^2)\cos(\alpha)-2ar)|}$,
- $OJ = \dfrac{|r\sin(\beta)(b^2-r^2)|}{|\cos(\alpha)\sin(\beta)(b^2-r^2)-\sin(\alpha)((r^2+b^2)\cos(\beta)-2br)|}$,
- $CI = \dfrac{r|\sin(\beta)|(a^2+r^2-2ar\cos(\alpha))}{|\cos(\beta)\sin(\alpha)(a^2-r^2)-\sin(\beta)((r^2+a^2)\cos(\alpha)-2ar)|}$,
- $CJ = \dfrac{r|\sin(\alpha)|(b^2+r^2-2br\cos(\beta))}{|\cos(\alpha)\sin(\beta)(b^2-r^2)-\sin(\alpha)((r^2+b^2)\cos(\beta)-2br)|}$,
- $\dfrac{CB'}{OB'}\cdot\dfrac{OI}{CI} = \dfrac{CJ}{OJ}\cdot\dfrac{OA'}{CA'}$ *(see also* [2] *page 63).*

Proof. A straight application of previous calculations. □

For any reals α, β, r and a we define $F^{\pm}_{\alpha,\beta}(a)$ whenever the denominator is non zero by:

$$F^{\pm}_{\alpha,\beta}(a) = \frac{r\sin\alpha(a^2-r^2)\pm r\sin\beta(a^2+r^2-2ar\cos(\alpha))}{\cos(\beta)\sin(\alpha)(a^2-r^2)-\sin(\beta)((r^2+a^2)\cos(\alpha)-2ar)}, \tag{6}$$

(with the same sign for $\pm$ on both sides). By Lemma 2.1 we know that

$$F^{+}_{\alpha,\beta}(a) = \frac{r(a\sin(s)+r\sin(d))}{a\sin(d)+r\sin(s)} \quad \text{and} \quad F^{-}_{\alpha,\beta}(a) = \frac{r(a\cos(s)-r\cos(d))}{a\cos(d)-r\cos(s)}.$$

In order to get identities of the form (1) we need to solve in b, $F^{\pm}_{\alpha,\beta}(a) = \pm F^{\pm}_{\beta,\alpha}(b)$. This is summarizd in the next theorems.

Theorem 2.1. *Under previous notation assuming* $\sin(\alpha)\sin(\beta)\neq 0$*:*

(1) $F^{+}_{\alpha,\beta}(a) = F^{+}_{\beta,\alpha}(b)$ *if and only if*

$$b = r\frac{a(1-\cos(\alpha)\cos(\beta))+r(\cos(\beta)-\cos(\alpha))}{r(1-\cos(\alpha)\cos(\beta))+a(\cos(\beta)-\cos(\alpha))}.$$

(2) $F^+_{\alpha,\beta}(a) = -F^+_{\beta,\alpha}(b)$ *if and only if* $b = -a$.

(3) $F^+_{\alpha,\beta}(a) = F^-_{\beta,\alpha}(b)$ *if and only if* $b = r\dfrac{a\cos(\alpha) - r}{a - r\cos(\alpha)}$.

(4) $F^+_{\alpha,\beta}(a) = -F^-_{\beta,\alpha}(b)$ *if and only if* $b = r\dfrac{a\cos(\beta) + r}{r\cos(\beta) + a}$.

(5) $F^-_{\alpha,\beta}(a) = F^-_{\beta,\alpha}(b)$ *if and only if* $b = a$.

(6) $F^-_{\alpha,\beta}(a) = -F^-_{\beta,\alpha}(b)$ *if and only if*

$$b = r\frac{a(1 + \cos(\alpha)\cos(\beta)) - r(\cos(\beta) + \cos(\alpha))}{a(\cos(\beta) + \cos(\alpha)) - r(1 + \cos(\alpha)\cos(\beta))}.$$

(7) $F^-_{\alpha,\beta}(a) = F^+_{\beta,\alpha}(b)$ *if and only if* $b = r\dfrac{r - a\cos(\beta)}{r\cos(\beta) - a}$.

(8) $F^-_{\alpha,\beta}(a) = -F^+_{\beta,\alpha}(b)$ *if and only if* $b = r\dfrac{r - a\cos(\alpha)}{a - r\cos(\alpha)}$.

Theorem 2.2. *Take* α *and* β *in* $]0;\pi[$, $a \geq 0$ *with* $r > 0$*:*

1) For $b = r\dfrac{a(1 - \cos(\alpha)\cos(\beta)) + r(\cos(\beta) - \cos(\alpha))}{r(1 - \cos(\alpha)\cos(\beta)) + a(\cos(\beta) - \cos(\alpha))}$,

- *if* $\alpha \geq \beta$ *then*
  - *for* $a \leq r$*, we have* $b \leq r$ *and* $|IO - IC| = |JO - JC|$,
  - *for* $a \geq r$*, we have* $b \geq r$ *and* $IO + IC = JO + JC$,
- *if* $\alpha < \beta$ *then*
  - *for* $\frac{r(\cos(\alpha)-\cos(\beta))}{1-\cos(\alpha)\cos(\beta)} \leq a \leq r$*, we have* $0 \leq b \leq r$ *and* $|IO - IC| = |JO - JC|$,
  - *for* $r \leq a < \frac{r(1-\cos(\alpha)\cos(\beta))}{(\cos(\alpha)-\cos(\beta))}$*, we have* $b \geq r$ *and* $IO + IC = JO + JC$.

2) For $b = \dfrac{r(a\cos(\alpha) - r)}{a - r\cos(\alpha)}$,

- *if* $0 < \alpha < \frac{\pi}{2}$ *then*
  - *for* $0 \leq a < r\cos(\alpha) \leq r$*, we have* $b \geq r$ *and* $|IO - IC| = |JO - JC|$,
  - *for* $a \geq \frac{r}{\cos(\alpha)} \geq r$*, we have* $b \leq r$ *and* $IO + IC = JO + JC$.

3) For $b = \dfrac{r(a\cos(\beta) + r)}{a + r\cos(\beta)}$,

- *if* $0 < \beta \leq \frac{\pi}{2}$ *then*
  - *for* $a \geq r$*, we have* $b \leq r$ *and* $IO + IC = JO + JC$,
  - *for* $a \leq r$*, we have* $b \geq r$ *and* $|IO - IC| = |JO - JC|$,
- *if* $\frac{\pi}{2} < \beta < \pi$ *then*
  - *for* $-r\cos(\beta) < a \leq r$*, we have* $b \geq r$ *and* $|IO - IC| = |JO - JC|$,
  - *for* $r \leq a \leq \frac{-r}{\cos(\beta)}$*, we have* $b \leq r$ *and* $IO + IC = JO + JC$.

4) For $b = a$,

- *for* $a \geq r$*, we have* $|IO - IC| = |JO - JC|$,
- *for* $a \leq r$*, we have* $IO + IC = JO + JC$.

5) For $b = r\dfrac{a(1+\cos(\alpha)\cos(\beta)) - r(\cos(\beta)+\cos(\alpha))}{a(\cos(\beta)+\cos(\alpha)) - r(1+\cos(\alpha)\cos(\beta))}$,

- *if* $\alpha + \beta \leq \pi$ *then*
  - *for* $0 \leq a \leq \frac{r(\cos(\alpha)+\cos(\beta))}{1+\cos(\alpha)\cos(\beta)} \leq r$, *we have* $b \leq r$ *and* $IO + IC = JO + JC$,
  - *for* $r \leq \frac{r(1+\cos(\alpha)\cos(\beta))}{\cos(\alpha)+\cos(\beta)} < a$, *we have* $b \geq r$ *and* $|IO - IC| = |JO - JC|$.

6) For $b = \dfrac{r(r - a\cos(\beta))}{r\cos(\beta) - a}$,

- *if* $0 < \beta < \frac{\pi}{2}$ *then*
  - *for* $0 \leq a < r\cos(\beta) \leq r$, *we have* $b \geq r$ *and* $IO + IC = JO + JC$,
  - *for* $a \geq \frac{r}{\cos(\beta)} \geq r$, *we have* $b \leq r$ *and* $|IO - IC| = |JO - JC|$.

7) For $b = \dfrac{r(r - a\cos(\alpha))}{a - r\cos(\alpha)}$,

- *if* $0 < \alpha < \frac{\pi}{2}$ *then*
  - *for* $r\cos(\alpha) < a \leq r$, *we have* $b \geq r$ *and* $IO + IC = JO + JC$,
  - *for* $r \leq a \leq \frac{r}{\cos(\alpha)}$, *we have* $b \leq r$ *and* $|IO - IC| = |JO - JC|$,
- *if* $\frac{\pi}{2} \leq \alpha < \pi$ *then*
  - *for* $a \geq r$, *we have* $b \leq r$ *and* $|IO - IC| = |JO - JC|$,
  - *for* $a \leq r$, *we have* $b \geq r$ *and* $IO + IC = JO + JC$.

Proof. The expressions of b are direct computations from Theorem 2.1. The bounds on (α, β, a) are to verify that b is nonnegative and the bounds on b can be deduced by taking the derivative of b as a function in a. □

The case $\alpha \in]0;\pi[$ and $\beta \in]-\pi;0[$ can be deduced from the previous theorem by replacing (verbatim) β with $|\beta|$ and switching identities, that is, $IO + IC = JO + JC \leftrightarrow |IO - IC| = |JO - JC|$ and vice versa. This gives for example Theorem 1.1 as a consequence of case *4)* which also appeared in [5].

We point out at last that if the coordinates of J are taken as real functions, the following pair values of b give the same point J:

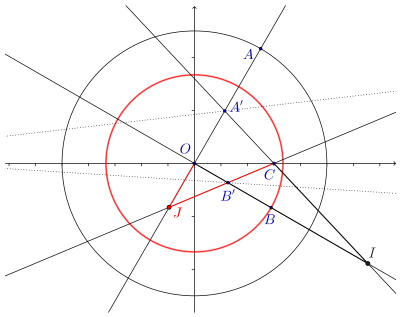
$$\left(a, r\frac{r - a\cos\beta}{r\cos(\beta) - a}\right), \left(-a, r\frac{r + a\cos\beta}{r\cos(\beta) + a}\right),$$

$$\left(r\frac{a\cos(\alpha) - r}{a - r\cos(\alpha)}, r\frac{a(1-\cos(\alpha)\cos(\beta)) + r(\cos(\beta)-\cos(\alpha))}{r(1-\cos(\alpha)\cos(\beta)) + a(\cos(\beta)-\cos(\alpha))}\right),$$

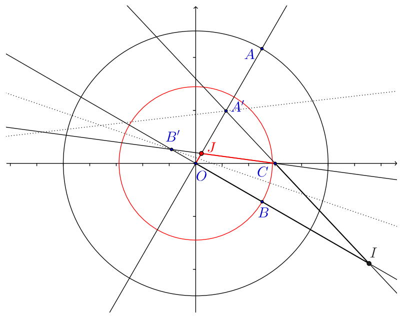
and

$$\left(r\frac{r - a\cos(\alpha)}{a - r\cos(\alpha)}, r\frac{a(1+\cos(\alpha)\cos(\beta)) - r(\cos(\beta)+\cos(\alpha))}{a(\cos(\beta)+\cos(\alpha)) - r(1+\cos(\alpha)\cos(\beta))}\right).$$

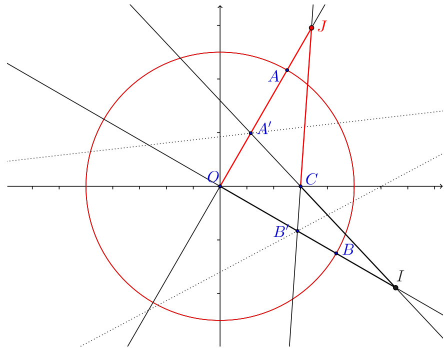
Example 2.1. *In this example we illustrate the proven identities for* $a = 5$, $r = 3$, $\alpha = \frac{\pi}{3}$, *and* $\beta = -\frac{\pi}{6}$.



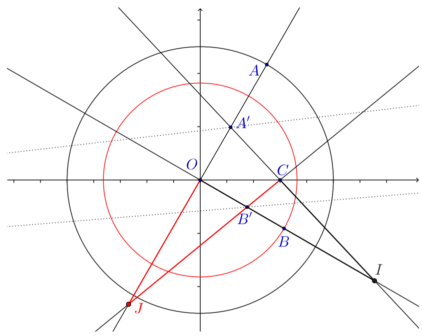
1) $b = \frac{288\sqrt{3}-21}{143}$, $|IO - IC| = |JO - JC|$



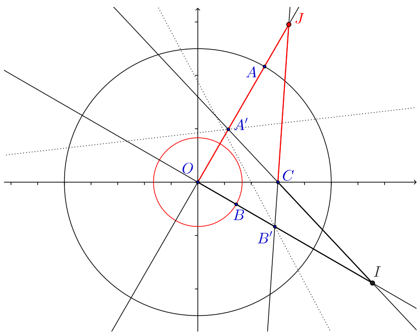
3) $b = \frac{96\sqrt{3}+45}{73}$, $|IO - IC| = |JO - JC|$



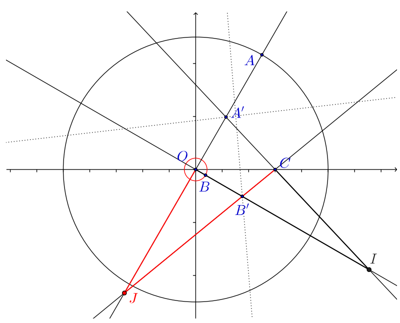
4) $b = 5$, $IO + IC = JO + JC$



5) $b = \frac{288\sqrt{3}+21}{143}$, $IO + IC = JO + JC$



6) $b = \frac{96\sqrt{3}-45}{73}$, $IO + IC = JO + JC$



7) $b = \frac{3}{7}$, $IO + IC = JO + JC$.

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