International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 8, 2023, pp.38-41
Received August 30, 2023. Published on-line September 30, 2023.
web: http://www.journal-1.eu/
©The Author(s) This article is published with open access[1].

Intersections of circles centered at triangle vertices passing through the other vertices

Benjamin Lee Warren[a] [2]
University of Binghamton,
4400 Vestel Pkwy E, Binghamton, NY, 13902
e-mail: warrenbenjamin52@yahoo.com

Abstract. For a given triangle, draw a circle which is centered at that vertex and passes through another vertex. Do this for all the vertices. We prove two results about when three of the circles intersect at a point.

Mathematics Subject Classification (2010). 51N20

1. Introduction

Many results exist regarding when certain circles in a given geometric construction all meet at a point (such as Miquel's Theorem [1]). Here, we suppose that for a given triangle, centered around each vertex are two circles which respectively each pass through the other vertices. In such a construction, it turns out that if a certain three of these circles meet at a point, then so do the other three circles.

2. Preliminaries

Let a, b, and c be three non-collinear points in a plane.
Let $\mathcal{O}Ab$ be the circle centered at a that passes through b.
Let $\mathcal{O}Ac$ be the circle centered at a that passes through c.
Let $\mathcal{O}Ba$ be the circle centered at b that passes through a.
Let $\mathcal{O}Bc$ be the circle centered at b that passes through c.
Let $\mathcal{O}Ca$ be the circle centered at c that passes through a.
Let $\mathcal{O}Cb$ be the circle centered at c that passes through b.

[1]This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.
[2]Corresponding author

3. Results

We have the following two results.
1. $\mathcal{O}Cb$ intersects a point where $\mathcal{O}Ab$ and $\mathcal{O}Ba$ intersect if and only if $\mathcal{O}Ca$ intersects a point where $\mathcal{O}Ac$ and $\mathcal{O}Bc$ intersect, and also
2. $\mathcal{O}Ca$ intersects a point where $\mathcal{O}Ab$ and $\mathcal{O}Ba$ intersect if and only if $\mathcal{O}Cb$ intersects a point where $\mathcal{O}Ac$ and $\mathcal{O}Bc$ intersect.

Proof. By translations, rotations, reflection, and symmetry, we will lose no generality by taking a to have Cartesian coordinates $(x_a, 0)$, b to have coordinates $(x_b, 0)$ (where $x_a < x_b$), and c to have coordinates (x_c, y_c) where $x_b < x_c$ and $y_c > 0$.
1. Suppose that $\mathcal{O}Cb$ intersects a point where $\mathcal{O}Ab$ and $\mathcal{O}Ba$ intersect. There are two points where $\mathcal{O}Ab$ and $\mathcal{O}Ba$ intersect, namely the points $(\frac{x_a+x_b}{2}, \frac{\sqrt{3}}{2}(x_b - x_a))$ and
$(\frac{x_a+x_b}{2}, -\frac{\sqrt{3}}{2}(x_b - x_a))$. Since y_c is positive, suppose that $\mathcal{O}Cb$ intersects the point $(\frac{x_a+x_b}{2}, \frac{\sqrt{3}}{2}(x_b - x_a))$.
$\mathcal{O}Cb$ is given by the equation $(x - x_c)^2 + (y - y_c)^2 = (x_c - x_b)^2 + y_c^2$, so if it intersects the point $(\frac{x_a+x_b}{2}, \frac{\sqrt{3}}{2}(x_b - x_a))$ then x_c and y_c must be constrained by the condition
$(\frac{x_a+x_b}{2} - x_c)^2 + (\frac{\sqrt{3}}{2}(x_b - x_a) - y_c)^2 = (x_c - x_b)^2 + y_c^2$
$\iff (x_a - x_b)(x_a - x_c + \sqrt{3}y_c) = 0 \iff y_c = \frac{x_c - x_a}{\sqrt{3}}$.
$\mathcal{O}Ac$ and $\mathcal{O}Bc$ intersect at two points, (x_c, y_c) and $(x_c, -y_c)$. The distance between the points (x_c, y_c) and $(x_c, -y_c)$ is $2y_c$. The distance from the point $(x_c, -y_c)$ to a is $\sqrt{(x_c - x_a)^2 + y_c^2} = \sqrt{3y_c^2 + y_c^2} = 2y_c$ so $\mathcal{O}Ca$ intersects the point $(x_c, -y_c)$.
Conversely, suppose that $\mathcal{O}Ca$ intersects a point where $\mathcal{O}Ac$ and $\mathcal{O}Bc$ intersect. This point is $(x_c, -y_c)$ (since the only other one is c but c does not pass through $\mathcal{O}Ca$). $\mathcal{O}Ca$ is given by the equation $(x - x_c)^2 + (y - y_c)^2 = (x_c - x_a)^2 + y_c^2$, so if it passes through the point $(x_c, -y_c)$ then x_c and y_c must be constrained by the condition $(-2y_c)^2 = (x_c - x_a)^2 + y_c^2 \iff (x_c - x_a)^2 = 3y_c^2$.
The distance from b to c is $\sqrt{(x_c - x_b)^2 + y_c^2}$. The distance from c to the point $(\frac{x_a+x_b}{2}, \frac{\sqrt{3}}{2}(x_b - x_a))$ is $\sqrt{(\frac{x_a+x_b}{2} - x_c)^2 + (\frac{\sqrt{3}}{2}(x_b - x_a) - y_c)^2} =$
$\sqrt{x_c^2 + x_a^2 - x_a x_b - x_a x_c + y_c^2 + x_b^2 - x_b x_c - \sqrt{3}x_b y_c + \sqrt{3}x_a y_c} =$
$(x_c^2 + (x_c - \sqrt{3}y_c)^2 - (x_c - \sqrt{3}y_c)x_b - (x_c - \sqrt{3}y_c)x_c + y_c^2 + x_b^2 - x_b x_c - \sqrt{3}x_b y_c + \sqrt{3}(x_c - \sqrt{3}y_c)y_c)^{\frac{1}{2}}$
$= \sqrt{(x_c - x_b)^2 + y_c^2}$ so $\mathcal{O}Cb$ intersects the point $(\frac{x_a+x_b}{2}, \frac{\sqrt{3}}{2}(x_b - x_a))$.
2. Suppose that $\mathcal{O}Ca$ intersects a point where $\mathcal{O}Ab$ and $\mathcal{O}Ba$ intersect. Since y_c is positive, suppose that $\mathcal{O}Ca$ intersects the point $(\frac{x_a+x_b}{2}, -\frac{\sqrt{3}}{2}(x_b - x_a))$.
$\mathcal{O}Ca$ is given by the equation $(x - x_c)^2 + (y - y_c)^2 = (x_c - x_a)^2 + y_c^2$, so if it intersects the point $(\frac{x_a+x_b}{2}, -\frac{\sqrt{3}}{2}(x_b - x_a))$ then x_c and y_c must be constrained by the condition
$(\frac{x_a+x_b}{2} - x_c)^2 + (-\frac{\sqrt{3}}{2}(x_b - x_a) - y_c)^2 = (x_c - x_a)^2 + y_c^2$
$\iff -(x_a - x_b)(x_b - x_c + \sqrt{3}y_c) = 0 \iff y_c = \frac{x_c - x_b}{\sqrt{3}}$. $\mathcal{O}Ac$ and $\mathcal{O}Bc$ intersect at two points, (x_c, y_c) and $(x_c, -y_c)$. The distance between the points (x_c, y_c) and $(x_c, -y_c)$ is $2y_c$. The distance from the point $(x_c, -y_c)$ to b is $\sqrt{(x_c - x_b)^2 + y_c^2} = \sqrt{3y_c^2 + y_c^2} = 2y_c$ so $\mathcal{O}Cb$ intersects the point $(x_c, -y_c)$.

Conversely, suppose that $\mathcal{O}Cb$ intersects the point $(x_c, -y_c)$. $\mathcal{O}Cb$ is given by the equation $(x - x_c)^2 + (y - y_c)^2 = (x_c - x_b)^2 + y_c^2$, so if it passes through the point $(x_c, -y_c)$ then x_c and y_c must be constrained by the condition $(-2y_c)^2 = (x_c - x_b)^2 + y_c^2 \iff (x_c - x_b)^2 = 3y_c^2$.
The distance from a to c is $\sqrt{(x_c - x_a)^2 + y_c^2}$. The distance from c to the point $(\frac{x_a+x_b}{2}, -\frac{\sqrt{3}}{2}(x_b - x_a))$ is $\sqrt{(\frac{x_a+x_b}{2} - x_c)^2 + (-\frac{\sqrt{3}}{2}(x_b - x_a) - y_c)^2}$
$\sqrt{(\frac{x_a+(x_c-\sqrt{3}y_c))}{2} - x_c)^2 + (-\frac{\sqrt{3}}{2}((x_c - \sqrt{3}y_c) - x_a) - y_c)^2}$
$= \sqrt{(x_c - x_a)^2 + y_c^2}$ so $\mathcal{O}Ca$ intersects the point $(\frac{x_a+x_b}{2}, -\frac{\sqrt{3}}{2}(x_b - x_a))$.

□

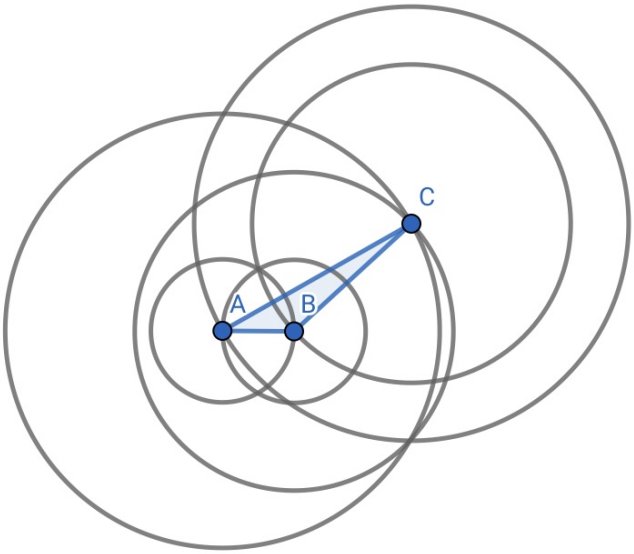

Figure 1. $\mathcal{O}Cb$ intersects a point where $\mathcal{O}Ab$ and $\mathcal{O}Ba$ intersect if and only if $\mathcal{O}Ca$ intersects a point where $\mathcal{O}Ac$ and $\mathcal{O}Bc$ intersect.

References

[1] Weisstein, Eric W. "Miquel's Theorem." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/MiquelsTheorem.html.

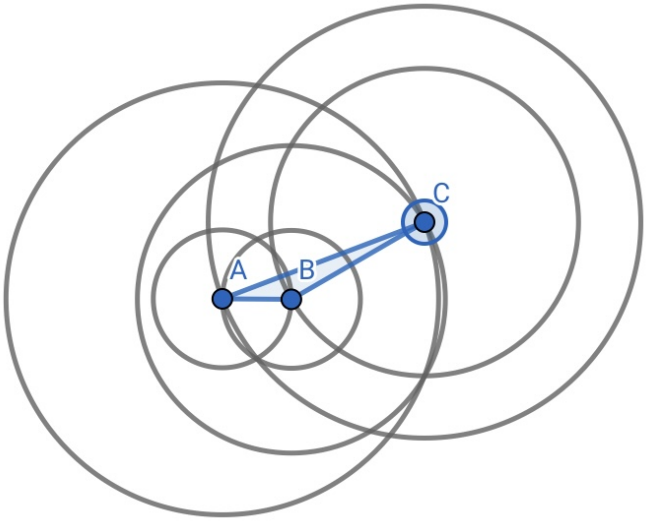

Figure 2. $\mathcal{O}Ca$ intersects a point where $\mathcal{O}Ab$ and $\mathcal{O}Ba$ intersect if and only if $\mathcal{O}Cb$ intersects a point where $\mathcal{O}Ac$ and $\mathcal{O}Bc$ intersect.