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Equivalent conditions for a triangle to be acute, right, or obtuse

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Abstract. We determine whether a triangle is acute, right, or obtuse based on information about the incenter and an excenter.

Keywords. incenter, excenter.

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1. Introduction

It is not an uncommon problem to be presented with limiting information about a triangle and then determining the triangle type. Suppose we were told to deduce whether triangle ΔABC is acute, right, or obtuse, by only being given an excenter and the incenter of ΔABC. If it turns out that that particular excenter is closer to some vertex of ΔABC than the incenter is to that vertex, then we instantly conclude that ΔABC is obtuse. Also, if it turns out that that excenter is equidistant from some vertex of ΔABC as the incenter is from that vertex, then we can instantly conclude that ΔABC is a right triangle. We need to know similar information about at least one more excenter before we can conclude whether ΔABC is acute, however. The proof offered here uses both analytic and synthetic geometry tools.

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2. Results

Theorem 2.1. *Let A, B, and C be the vertices of any triangle.*
1. ΔABC is acute $\iff$ two of the excenters of ΔABC each lie further from two vertices than the incenter does to either of those vertices.
2. ΔABC is a right triangle $\iff$ two of the excenters each lie the same distance from two vertices as the incenter does to either of those vertices.
3. ΔABC is obtuse $\iff$ two of the excenters of ΔABC each lie closer to two vertices than the incenter does to either of those vertices.

Proof. Without loss of generality, we let the cartesian coordinates of A, B, and C be given by $(0,0),(x_B,0)$, and (x_C,y_C) respectively where $0 < x_B$, $0 < y_C$, and $\Omega x_B < x_C$ where $(\Omega x_B, y_C)$ makes a right angle to points A and B.
The coordinates of the incenter, D, are
$(\frac{x_B\sqrt{x_C^2+y_C^2}+x_Bx_C)}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B}, \frac{x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})$.
The coordinates of two of the excenters are given by
$Q_1 = (\frac{x_B\sqrt{x_C^2+y_C^2}-x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B}, \frac{-x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B})$, and
$Q_2 = (\frac{x_B\sqrt{x_C^2+y_C^2}+x_Bx_C}{-\sqrt{(x_C-x_B)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B}, \frac{x_By_C}{-\sqrt{(x_C-x_B)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})$.
The distance from A to D is
$\sqrt{(\frac{x_B\sqrt{x_C^2+y_C^2}+x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2+(\frac{x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2}$.
The distance from C to D is
$\sqrt{(x_C-\frac{x_B\sqrt{x_C^2+y_C^2}+x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2+(y_C-\frac{x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2}$.
The distance from A to Q_1 is
$\sqrt{(\frac{x_B\sqrt{x_C^2+y_C^2}-x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B})^2+(\frac{-x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B})^2}$.
The distance from C to Q_2 is
$\sqrt{(x_C-\frac{x_B\sqrt{x_C^2+y_C^2}+x_Bx_C}{-\sqrt{(x_C-x_B)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2+(y_C-\frac{x_Bx_C}{-\sqrt{(x_C-x_B)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2}$.

A key tool that we will utilize is the fact that the incenter of any given triangle ΔABC is also the orthocenter of the excentral triangle of ΔABC. This means that since $\angle ADQ_1 = \angle CDQ_2$, and both $\angle DAQ_1$ and $\angle DCQ_2$ are right angles, it follows that ΔADQ_1 and ΔCDQ_2 are similar triangles. In particular, $AQ_1 : AD = CQ_2 : CD$, so one excenter of ΔABC being closer to a vertex than the incenter is to that vertex is enough to show that another excenter also has the same property to a different vertex. Similarly, one excenter of ΔABC being further from a vertex than the incenter is to that vertex is enough to show that another excenter also has the same property to a different vertex.

Case 1: ΔABC is acute. This is equivalent to $\Omega x_B < x_C < x_B \iff x_C = x_B - \varepsilon$ for some $0 < \varepsilon < x_B - \Omega x_B$. The distance from A to D is
$\sqrt{(\frac{x_B\sqrt{(x_B-\varepsilon)^2+y_C^2}+x_B(x_B-\varepsilon)}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}+x_B})^2+(\frac{x_By_C}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}+x_B})^2}$.

The distance from A to Q_1 is $\sqrt{(\frac{x_B\sqrt{(x_B-\varepsilon)^2+y_C^2}-x_B(x_B-\varepsilon)}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}-x_B})^2+(\frac{-x_By_C}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}-x_B})^2}$.
Let $f(\varepsilon)=(\frac{x_B\sqrt{(x_B-\varepsilon)^2+y_C^2}-x_B(x_B-\varepsilon)}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}-x_B})^2+(\frac{-x_By_C}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}-x_B})^2$
$-(\frac{x_B\sqrt{(x_B-\varepsilon)^2+y_C^2}+x_B(x_B-\varepsilon)}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}+x_B})^2-(\frac{x_By_C}{\sqrt{\varepsilon^2+y_C^2}+\sqrt{(x_B-\varepsilon)^2+y_C^2}+x_B})^2$. $f(\varepsilon)$ has exactly one root, which is at $\varepsilon=0$. We calculate $f'(0)=-2(\sqrt{a^2+b^2}-\frac{a^2}{b}-b)$ where $a=x_B$ and $b=y_C$. This is positive whenever a and b are positive since $a^4+a^2b^2>0 \iff a^4+2a^2b^2+b^4>a^2b^2+b^4 \iff (a^2+b^2)^2>b^2(a^2+b^2) \iff a^2+b^2>b\sqrt{a^2+b^2}$. Therefore, $f(\varepsilon)$ is positive when $0<\varepsilon<x_B-\Omega x_B$, which means that the distance from A to Q_1 is greater than the distance from A to D.
By the paragraph preceding case 1, we also conclude that the distance from C to Q_2 is greater than the distance from C to D.

Case 2: ΔABC is a right triangle. This is equivalent to $x_B=x_C$. The distance from A to D is $\sqrt{(\frac{x_B\sqrt{x_B^2+y_C^2}+x_B^2}{y_C+\sqrt{x_B^2+y_C^2}+x_B})^2+(\frac{x_By_C}{y_C+\sqrt{x_B^2+y_C^2}+x_B})^2}$, and the distance from A to Q_1 is $\sqrt{(\frac{x_B\sqrt{x_B^2+y_C^2}-x_B^2}{y_C+\sqrt{x_B^2+y_C^2}-x_B})^2+(\frac{-x_By_C}{y_C+\sqrt{x_B^2+y_C^2}-x_B})^2}$. Both distances are equal due to the identity
(1)
$$(\frac{a\sqrt{a^2+b^2}+a^2}{a+b+\sqrt{a^2+b^2}})^2+(\frac{ab}{a+b+\sqrt{a^2+b^2}})^2=(\frac{a\sqrt{a^2+b^2}-a^2}{\sqrt{a^2+b^2}-a+b})^2+(\frac{ab}{\sqrt{a^2+b^2}-a+b})^2$$
for any $a,b\in\mathbb{R}$ where $a+b\sqrt{a^2+b^2}=0$ and $\sqrt{a^2+b^2}-a+b=0$.
Also, the distance from C to D is $\sqrt{(x_B-\frac{x_B\sqrt{x_B^2+y_C^2}+x_B^2}{y_C+\sqrt{x_B^2+y_C^2}+x_B})^2+(y_C-\frac{x_By_C}{y_C+\sqrt{x_B^2+y_C^2}+x_B})^2}$, and the distance from C to Q_2 is $\sqrt{(x_B-\frac{x_B\sqrt{x_B^2+y_C^2}+x_B^2}{-y_C+\sqrt{x_B^2+y_C^2}+x_B})^2+(y_C-\frac{x_By_C}{-y_C+\sqrt{x_B^2+y_C^2}+x_B})^2}$. Both distances are equal due to the identity
(2)
$$(a-\frac{a\sqrt{a^2+b^2}+a^2}{a+b+\sqrt{a^2+b^2}})^2+(b-\frac{ab}{a+b+\sqrt{a^2+b^2}})^2=(a-\frac{a\sqrt{a^2+b^2}+a^2}{a-b+\sqrt{a^2+b^2}})^2+(b-\frac{ab}{a-b+\sqrt{a^2+b^2}})^2$$
for any $a,b\in\mathbb{R}$ where $a+b+\sqrt{a^2+b^2}=0$ and $a-b+\sqrt{a^2+b^2}=0$.

Case 3: ΔABC is obtuse. This is equivalent to $x_B<x_C \iff x_C=x_B-\varepsilon$ for $\varepsilon<0$. The same argument can be made as in case 1, with the main observation that since $f(\varepsilon)$ has one root, and here it is going in the negative direction, that indicates that $f(\varepsilon)$ is negative when $\varepsilon<0$, which means that the distance from A to Q_1 is less than the distance from A to D. By the paragraph preceding case 1, we again conclude that the distance from C to Q_2 is less than the distance from C to D. $\square$

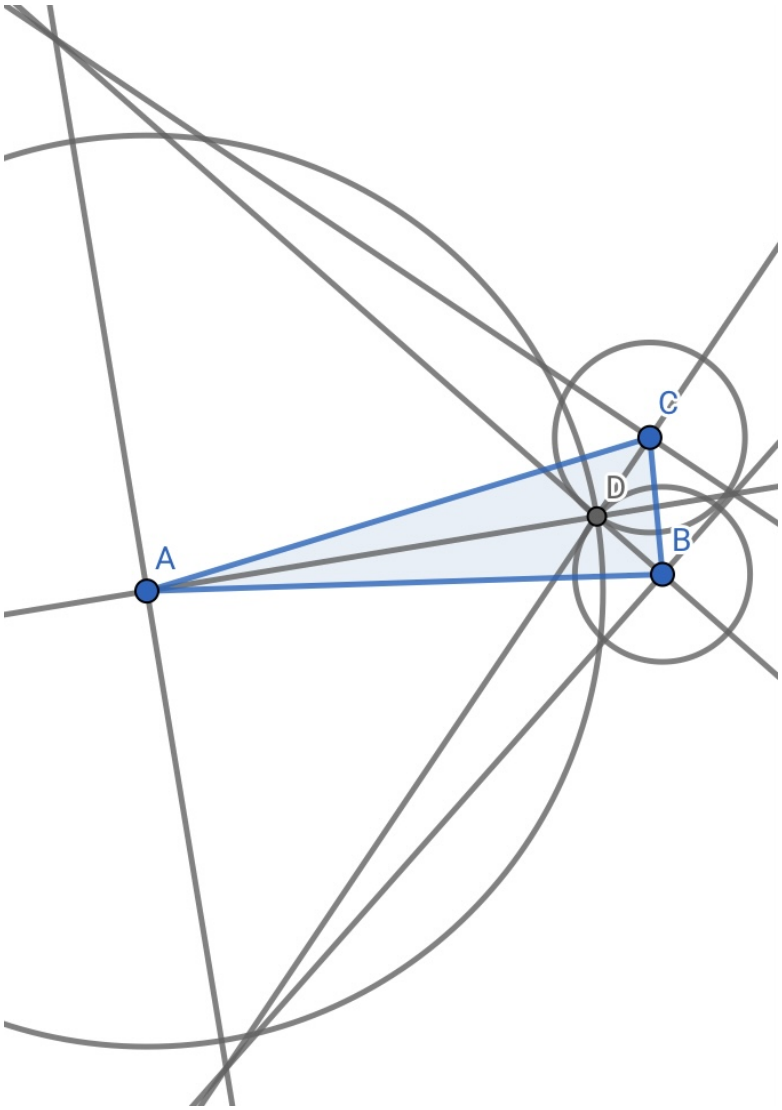

FIGURE 1. Two of the excenters of ΔABC lie further from A and C respectively than the incenter of ΔABC does to A and C respectively

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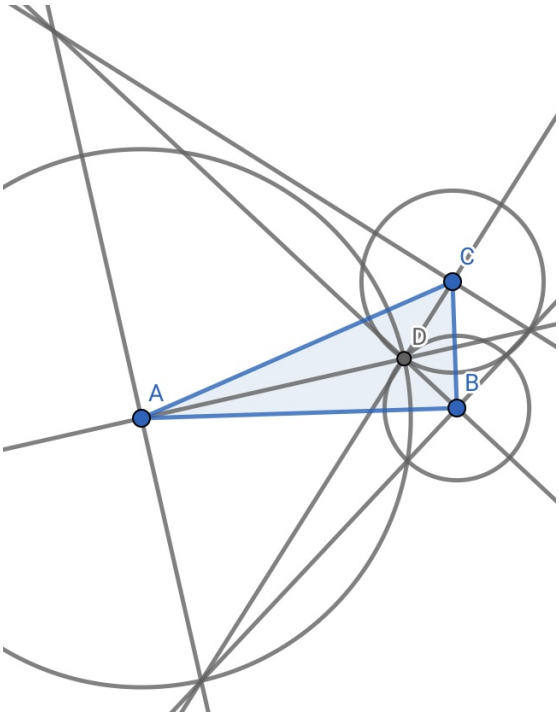

Figure 2. Two of the excenters of ΔABC are each equally distant from A and C respectively as the incenter of ΔABC is to A and C respectively.

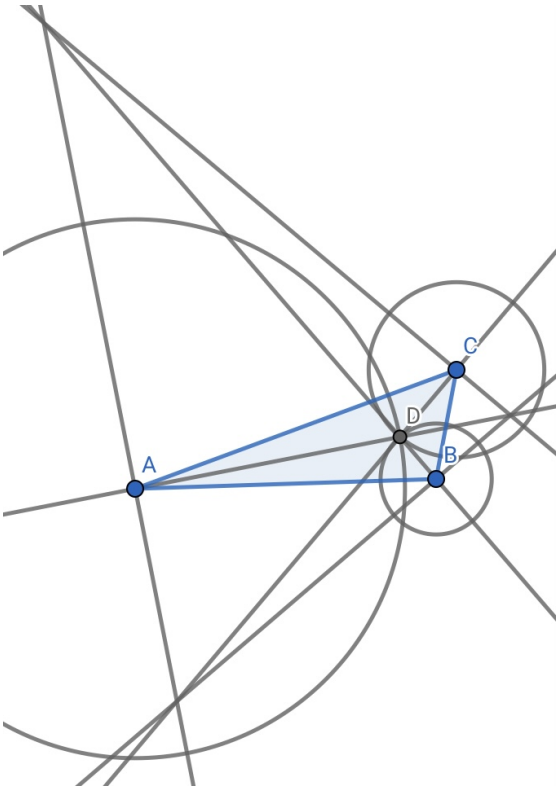

Figure 3. Two of the excenters of ΔABC lie closer to A and C respectively than the incenter of ΔABC does to A and C respectively.