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RADICAL AXES AND COMPLEX NUMBERS

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Abstract. The present paper considers a solution by complex numbers of a problem from the International Mathematical Olympiad in 2023. In the problem three points are chosen in an equilateral triangle in a special way. They define three circles with two common points. It is discussed a variant of the configuration in the case when one of the points is outsde the equilateral triangle. Two basic assertions lead to triplets of radical axes in the plane of the equilateral triangle. The axes themselves form an equilateral triangle. An idea is presented to obtain analogous assertions which are connected with circumscribed ellipses of the same triangles. Some of the assirtions remain to be hypotheses for now.

Key words: equilateral trangle, circumscribed circle, radical axis, gravity center, circumscribed ellipse, GSP, Maple

1. Introduction. The last International Mathematical Olympiad took place in Japan in 2023. One of the problems in the paper concerns the formulation of the following assirtion:

Proposition 1. *Given is an equilateral triangle ABC. The points* A_1, B_1 *and* C_1 *are in the interior of the triangle satisfying the conditions* $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$ *and* $\sphericalangle BA_1C + \sphericalangle CB_1A + \sphericalangle AC_1B = 480°$. *Let* BC_1 *and* CB_1 *intersect in* A_2, CA_1 *and* AC_1 *– in* B_2, *while* AB_1 *and* BA_1 *– in* C_2. *If* $\triangle A_1B_1C_1$ *is scalen, then the circumcircles of the triangles* AA_1A_2, BB_1B_2 *and* CC_1C_2 *have two common points* (Fig. 1).

A synthetic solution of the problem is presented in [1]. What will be considered here is an analytic one using complex numbers. It contains some peculiarities:

1) It is not easier to prove it in comparison with the synthetic one since it is connected with many calculations carried out by the program Maple;

2) Its idea is easier in the sense, that any special geometric considerations and additional constructions are not necessary concerning the configuration under study. The same idea works successfully in attacking other similar problems;

3) It gives possibilities to obtain additional properties of the equilateral triangle in connection with the same configuration;

4) It could be used to obtain a similar assertion when one of the points A_1, B_1 and C_1 is in the exterior of $\triangle ABC$.

5) Some of the results could be applied to other similar configurations.

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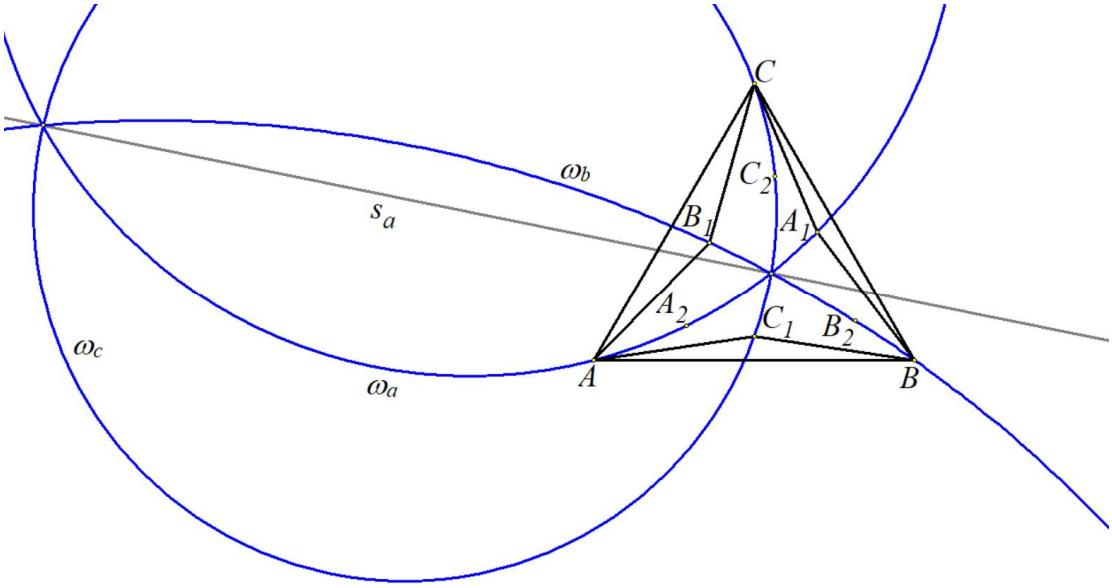

Fig. 1

2. Proof of proposition 1. Let $\mathbf{S}A_1BC = \mathbf{S}A_1CB = a$, $\mathbf{S}B_1CA = \mathbf{S}B_1AC = b$ and $\mathbf{S}C_1AB = \mathbf{S}C_1BA = g$ (Fig. 2). Since $\mathbf{S}BA_1C = 180° - 2a$, $\mathbf{S}CB_1A = 180° - 2b$ and $\mathbf{S}AC_1B = 180° - 2g$, then

$$\mathbf{S}BA_1C + \mathbf{S}CB_1A + \mathbf{S}AC_1B = 540° - 2(a+b+g).$$

It follows from this eqality and the conditon of the problem, that

Lemma 1. *The equality* $a+b+g = 30°$ *is fulfilled for the angles* a, b *and* g (Fig. 2).

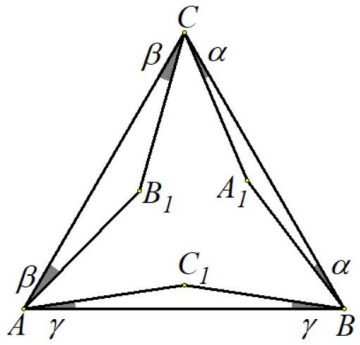

Fig. 2

It follows from lemma 1, that

$$g = \frac{p}{6} - a - b. \tag{1}$$

Consider a Gauss coordinate system with a center in the center of ΔABC. The point affixes will be denoted by the corresponding small letters. Additionally, we introduce the notations: $A_0 = e^{ia}$, $B_0 = e^{ib}$, $C_0 = \overline{A}_0\overline{B}_0 e^{i\frac{p}{6}}$.

Since $\overrightarrow{BA_1}$, $\overrightarrow{CB_1}$ and $\overrightarrow{AC_1}$ are obtained in rotating $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ respectively at angles α, β and $\gamma=\frac{\pi}{6}-\alpha-\beta$, the affixes of the points A_1, B_1 and C_1 fulfill the equalities:

$$a_1=b+(c-b)A_0,\ b_1=c+(a-c)B_0,\ c_1=a+(b-a)C_0. \tag{2}$$

If Z_1Z_2 is an arbitrary line in the plane of ΔABC, then its equation is the following:

$$(\bar{z}_1-\bar{z}_2)z-(z_1-z_2)\bar{z}-(\bar{z}_1z_2-z_1\bar{z}_2)=0. \tag{3}$$

From (2) and (3) we obtain

$$\begin{aligned} &BA_1: z+A_0^2bc\bar{z}-A_0^2c-b=0,\\ &CB_1: z+B_0^2ca\bar{z}-B_0^2a-c=0,\\ &AC_1: 2A_0^2B_0^2z+\left(1+i\sqrt{3}\right)ab\bar{z}-2A_0^2B_0^2a-\left(1+i\sqrt{3}\right)b=0. \end{aligned} \tag{4}$$

Concerning the equations of the mediatrices l_a, l_b and l_c with respect to the segments BC, CA and AB, respectively, it is deduced easily from the perpendicularity property, that:

$$l_a: z-bc\bar{z}=0,\ l_b: z-ca\bar{z}=0,\ l_c: z-ab\bar{z}=0. \tag{5}$$

Since $BA_1\cap l_a=A_1$, $CB_1\cap l_b=B_1$ and $AC_1\cap l_c=C_1$, we determine the corresponding systems of two equations with two unknowns by means of the equations (4) and (5). The affixes of the points A_1, B_1 and C_1 are:

$$a_1=\frac{A_0^2c+b}{A_0^2+1},\ b_1=\frac{B_0^2a+c}{B_0^2+1},\ c_1=\frac{2A_0^2B_0^2a+\left(1+i\sqrt{3}\right)b}{2A_0^2B_0^2+1+i\sqrt{3}}. \tag{6}$$

Now, from (3) and (6) we obtain the equations:

$$\begin{aligned} &BC_1: \left(1+i\sqrt{3}\right)z+2A_0^2B_0^2ab\bar{z}-2A_0^2B_0^2a-\left(1+i\sqrt{3}\right)b=0,\\ &CA_1: A_0^2z+bc\bar{z}-A_0^2c-b=0,\\ &AB_1: B_0^2z+ca\bar{z}-B_0^2a+c=0. \end{aligned} \tag{7}$$

Analogously, since $BC_1\cap CB_1=A_2$, $CA_1\cap AC_1=B_2$ and $AB_1\cap BA_1=C_2$, we determine the corresponding systems of two equations with two unknowns by means of the equations (4) and (7). The affixes of the points A_2, B_2 and C_2 are:

$$\begin{aligned} a_2&=\frac{2A_0^2B_0^2a(b-c)+\left(2A_0^2-1-i\sqrt{3}\right)bc}{2A_0^2b-\left(1+i\sqrt{3}\right)c},\\ b_2&=\frac{2A_0^2B_0^2ca-A_0^2ca\left(1+i\sqrt{3}\right)+b(c-a)\left(1+i\sqrt{3}\right)}{A_0^2\left[2B_0^2c-a\left(1+i\sqrt{3}\right)\right]},\\ c_2&=\frac{\left(A_0^2B_0^2-1\right)ab+A_0^2c(b-c)}{A_0^2B_0^2b-a}. \end{aligned} \tag{8}$$

Let the following equalities be fulfilled:

$$a=1,\ b=\omega=e^{i.\frac{2\pi}{3}},\ c=\bar{\omega}=e^{-i.\frac{2\pi}{3}}. \tag{9}$$

Note, that $1+\omega+\omega^2=1+\omega+\bar{\omega}=0$, which will be used during the next calculations.

The circumcircles w_a, w_b and w_c of the triangles AA_1A_2, BB_1B_2 and CC_1C_2, respectively, have the following equations:

$$w_a : z\bar{z} + \bar{p}_1 z + p_1\bar{z} + q_1 = 0,\ w_b : z\bar{z} + \bar{p}_2 z + p_2\bar{z} + q_2 = 0,\ w_c : z\bar{z} + \bar{p}_3 z + p_3\bar{z} + q_3 = 0.$$

Replacing z with a, a_1 and a_2 in the equation of w_a, leads to the system of three equatoins with three unknowns p_1, $\bar{p}_1$ and q:

$$a\bar{a} + \bar{p}_1 a + p_1\bar{a} + q_1 = 0,\ a_1\bar{a}_1 + \bar{p}_1 a_1 + p_1\bar{a}_1 + q_1 = 0,\ a_2\bar{a}_2 + \bar{p}_1 a_2 + p_1\bar{a}_2 + q_1 = 0.$$

From here and the equations (6), (8) and (9), the following formulae are obtained:

$$p_1 = -\frac{2A^4B^4 + 2A^2B^4 - A^2B^2\left(3 + i\sqrt{3}\right) - A^2\left(1 - i\sqrt{3}\right) + 2}{\left(2A^2B^4 - 1 - i\sqrt{3}\right)\left(A^2 + 1\right)},$$

$$\bar{p}_1 = \frac{\left(1 + i\sqrt{3}\right)\left[2A^4B^4 - A^2B^4\left(1 + i\sqrt{3}\right) - A^2B^2\left(3 - i\sqrt{3}\right) + 2\left(A^2 + 1\right)\right]}{2\left(2A^2B^4 - 1 - i\sqrt{3}\right)\left(A^2 + 1\right)}, \tag{10}$$

$$q_1 = -\frac{\left(1 + i\sqrt{3}\right)\left(2A^2 - 1 - i\sqrt{3}\right)}{4\left(A^2 + 1\right)}.$$

In a similar way, from the equations of w_b and w_c using the equalities (6), (8) and (9), the following formulae are obtained:

$$p_2 = -\frac{2A^4B^4 - A^4B^2\left(1 + i\sqrt{3}\right) - A^2B^2\left(3 - i\sqrt{3}\right) + 2\left(B^2 + 1\right)}{\left(2A^2B^4 - 1 - i\sqrt{3}\right)\left(B^2 + 1\right)},$$

$$\bar{p}_2 = \frac{\left(1 + i\sqrt{3}\right)\left[2A^4B^4 - 2A^4B^2 - A^2B^2\left(3 + i\sqrt{3}\right) - B^2\left(1 - i\sqrt{3}\right) + 2\right]}{2\left(2A^4B^2 - 1 - i\sqrt{3}\right)\left(B^2 + 1\right)}, \tag{11}$$

$$q_2 = -\frac{\left(1 + i\sqrt{3}\right)\left(2B^2 - 1 - i\sqrt{3}\right)}{4\left(B^2 + 1\right)},$$

$$p_3 = \frac{\left(1 + i\sqrt{3}\right)\left[2A^4B^2 + A^2B^4\left(1 - i\sqrt{3}\right) - A^2B^2\left(3 + i\sqrt{3}\right) + A^2\left(1 + i\sqrt{3}\right) - B^2\left(1 - i\sqrt{3}\right)\right]}{2\left(2A^2B^2 + 1 + i\sqrt{3}\right)\left(A^2 + B^2\right)},$$

$$\bar{p}_3 = \frac{\left(1 + i\sqrt{3}\right)\left[2A^4B^2 + A^2B^4\left(1 + i\sqrt{3}\right) - 2i\sqrt{3}A^2B^2 - 2A^2 - B^2\left(1 - i\sqrt{3}\right)\right]}{2\left(2A^2B^2 + 1 + i\sqrt{3}\right)\left(A^2 + B^2\right)}, \tag{12}$$

$$q_3 = -\frac{\left(1 - i\sqrt{3}\right)\left(A^2B^2 + 1\right)}{2A^2B^2 + 1 + i\sqrt{3}}.$$

Subtract the equations of w_b and w_c from the equation of w_a and deduce the next two equations of lines:

$$\left(\bar{p}_1 - \bar{p}_2\right)z - \left(p_1 - p_2\right)z - \left(q_1 - q_2\right) = 0,\ \left(\bar{p}_1 - \bar{p}_3\right)z - \left(p_1 - p_3\right)z - \left(q_1 - q_3\right) = 0. \tag{13}$$

Using the equations (10), (11) and (12) we obtain the equalities $\frac{\overline{p}_1 - \overline{p}_2}{\overline{p}_1 - \overline{p}_3} - \frac{p_1 - p_2}{p_1 - p_3} = 0$, $\frac{\overline{p}_1 - \overline{p}_2}{\overline{p}_1 - \overline{p}_3} - \frac{q_1 - q_2}{q_1 - q_3} = 0$, applying the calculating capacities of Maple. It follows from here, that the equations (13) present one and the same line s_a. Consequantly, the circles w_a, w_b and w_c belong to a bundle of circles with radical axis s_a. Thus, proposition 1 is proved.

3. An equilateral triangle, formed by radical axes. Instead of the considered configuration with three circles,which is obtained with concrete values of the angles a, b and g (Fig. 2), two triplets of circles could be constructed additionally using the same values of a, b and g. One of them is based on the equalities $\mathbf{S}A_1BC = \mathbf{S}A_1CB = g$, $\mathbf{S}B_1CA = \mathbf{S}B_1AC = a$ and $\mathbf{S}C_1AB = \mathbf{S}C_1BA = b$ (Fig. 3). The other one is based on the equalities $\mathbf{S}A_1BC = \mathbf{S}A_1CB = b$, $\mathbf{S}B_1CA = \mathbf{S}B_1AC = g$ and $\mathbf{S}C_1AB = \mathbf{S}C_1BA = a$ (Fig. 4). The corresponding configurations are obtained when the triangle ΔABC is rotated at $120°$ and $240°$, respecively, while the values of the angles a, b and g remain fixed.

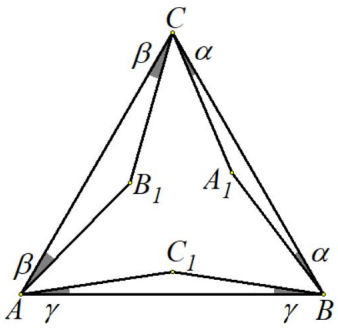

Fig. 2

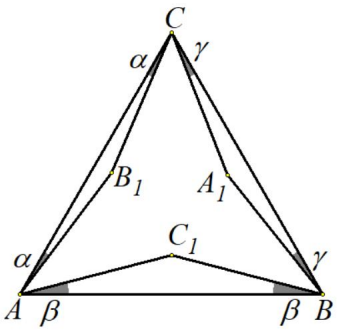

Fig. 3

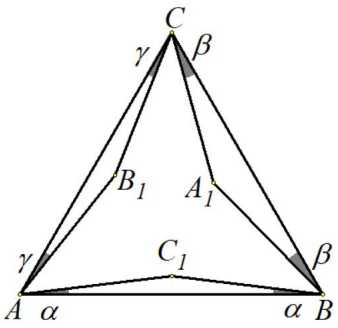

Fig. 4

Using the equalities (10) and after some transformations we obtain, that the radical axis s_a has the following equation:

(14) $$s_a : 2Uz + \left(1 - i\sqrt{3}\right)V\overline{z} + W = 0,$$

where

$$U = B^2\left[2A^6B^6 + A^6B^4\left(1 - i\sqrt{3}\right) - A^4B^4\left(1 - i\sqrt{3}\right) - A^6B^2\left(1 + i\sqrt{3}\right) - \right.$$
$$\left. - 2B^6 - A^2B^2\left(1 + i\sqrt{3}\right) - B^4\left(1 - i\sqrt{3}\right) + B^2\left(1 + i\sqrt{3}\right) + 2\right],$$

(15)
$$V = A^2\left[2A^6B^6 + A^4B^6\left(1 - i\sqrt{3}\right) - A^4B^4\left(1 - i\sqrt{3}\right) - A^2B^6\left(1 + i\sqrt{3}\right) - \right.$$
$$\left. - 2B^6 - A^2B^2\left(1 + i\sqrt{3}\right) - B^4\left(1 - i\sqrt{3}\right) + B^2\left(1 + i\sqrt{3}\right) + 2\right],$$
$$W = \left(A^2 - B^2\right)\left(2A^4B^2 - 1 - i\sqrt{3}\right)\left(2A^2B^4 - 1 - i\sqrt{3}\right).$$

The case $\mathbf{S}A_1BC = \mathbf{S}A_1CB = g$, $\mathbf{S}B_1CA = \mathbf{S}B_1AC = a$ and $\mathbf{S}C_1AB = \mathbf{S}C_1BA = b$ is realized when instead of the equations (9) we use the equalities:

$$a=\overline{w}=e^{-i.\frac{2p}{3}},\ b=1,\ c=w=e^{i.\frac{2p}{3}}. \tag{9'}$$

From here using the already derived formulae by means of Maple, we obtain, that the radical axis s_b of the cirles w_a, w_b and w_c has the followng equation:

$$s_b: 4Uz - 4V\overline{z} - \left(1 - i\sqrt{3}\right)W = 0, \tag{16}$$

where U, V and W are presented by the equalities (15).

The case $\mathbf{S}A_1BC = \mathbf{S}A_1CB = b$, $\mathbf{S}B_1CA = \mathbf{S}B_1AC = g$ and $\mathbf{S}C_1AB = \mathbf{S}C_1BA = a$ is realized when instead of the equalities (9) we use, that:

$$a=w=e^{i.\frac{2p}{3}},\ b=\overline{w}=e^{-i.\frac{2p}{3}},\ c=1. \tag{9''}$$

It follows, that the radical axis s_c of w_a, w_b and w_c has the following equation:

$$s_c: \left(1 - i\sqrt{3}\right)Uz + 2V\overline{z} - W = 0, \tag{17}$$

where U, V and W are expressed by the equalities (15).

Let $s_b \cap s_c = A_3$, $s_c \cap s_a = B_3$ and $s_a \cap s_b = C_3$. After solving the three systems of two equations each, derived by combining the equations (14), (16) and (17), we obtain by Maple, that

$$a_3 = -\frac{\left(1 - i\sqrt{3}\right)W}{4U},\ b_3 = \frac{W}{2U},\ c_3 = -\frac{\left(1 + i\sqrt{3}\right)W}{4U}, \tag{18}$$

where U, V and W are expressed by the equalities (15).

The following two results are obtained from (18):

$$A_3B_3^2 = B_3C_3^2 = C_3A_3^2 = \frac{3\left(1 + i\sqrt{3}\right)W^2}{8UV}, \tag{19}$$

$$a_3 + b_3 + c_3 = 0. \tag{20}$$

The equalities (19) and (20) show that the follwoing assirtion is fulfiled:

Proposition 2. *The triangle $A_3B_3C_3$ is equilateral and its center coinsides with the center of ΔABC* (Fig. 5).

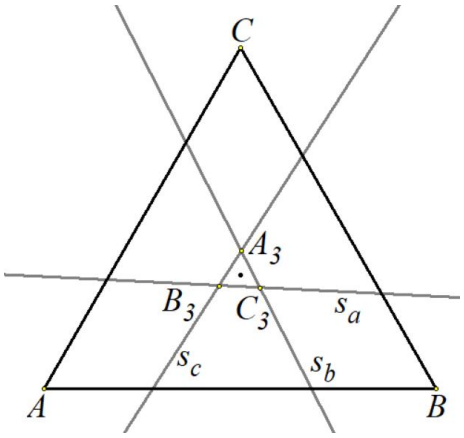

Fig. 5

4. The case of exterior point. The already proved proposition 1 puts forward the question for the consideration of a similar configuration, in which one of the points A_1, B_1 and C_1 is exterior for the equalateral triangle. It turns out, that a similar assirtion could be obtained when the condition $\mathbf{S}BA_1C+\mathbf{S}CB_1A+\mathbf{S}AC_1B=480^\circ$ is changed in a suitable way. Thus, we come to the following

Proposition 3. *Given is an equilateral triangle ABC. The points A_1 and B_1 are interior, while C_1 is exterior, and $BA_1=A_1C$, $CB_1=B_1A$, $AC_1=C_1B$ and $\mathbf{S}BA_1C+\mathbf{S}CB_1A-\mathbf{S}AC_1B=120^\circ$. Let BC_1 and CB_1 intersect in A_2, CA_1 and AC_1 interesct in B_2, while AB_1 and BA_1 intersect in C_2. If $\Delta A_1B_1C_1$ is scalen, then the sircumcircles of the triangles AA_1A_2, BB_1B_2 and CC_1C_2 have two common points* (Fig. 6).

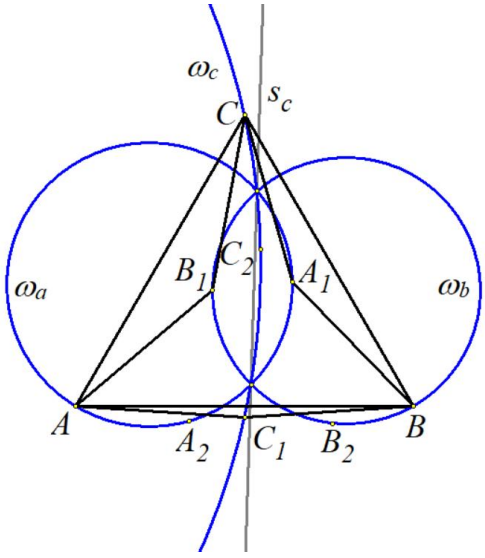

Fig. 6

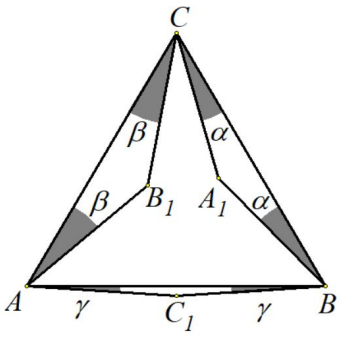

Fig. 7

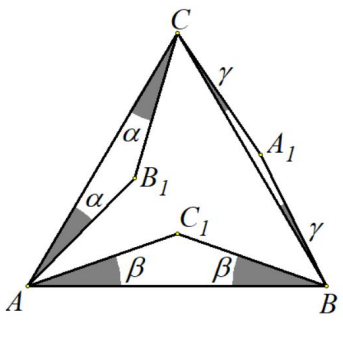

Fig. 8

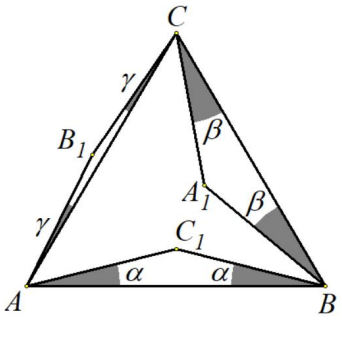

Fig. 9

As in proposition 1, denote the circumcircles of the triangles AA_1A_2, BB_1B_2 and CC_1C_2 with w_a, w_b and w_c (Fig. 6). Also, we introduce the notations $\mathbf{S}A_1BC=\mathbf{S}A_1CB=a$, $\mathbf{S}B_1CA=\mathbf{S}B_1AC=b$ and $\mathbf{S}C_1AB=\mathbf{S}C_1BA=g$ (Fig. 7). Analogously to lemma 1 we formulate the following assirtion:

Lemma 2. *The angles a, b and g fulfill the equality $a+b-g=30^\circ$* (Fig. 7).

Analogously to the proof of proposition 1, it follows from lemma 2, that the realization of the points A_1 and B_1 requires rotation of the vectors $\overrightarrow{BC}$ and $\overrightarrow{CA}$ at angles a and b respectively in counterclockwise direction. For the point C_1 it is necessary to rotate the vector $\overrightarrow{AB}$ at angle $g = a + b - \frac{p}{6}$ in clockwise direction. This means, that the vector $\overrightarrow{AB}$ should be rotated at angle $-g = \frac{p}{6} - a - b$ in counterclockwise direction. Consequently, all derived formulae in the complex plane in connection with the proof of proposition 1 could be used in the proof of propsition 2. In such a way we obtain justification of proposition 3 without any need to repeat the calculations from the proof of proposition 1.

Let the radical axis, determined with the circles from proposition 3, be s_c. Moreover, the triplets of circles obtained by the equalities $\mathbf{S}A_1BC = \mathbf{S}A_1CB = g$, $\mathbf{S}B_1CA = \mathbf{S}B_1AC = a$ and $\mathbf{S}C_1AB = \mathbf{S}C_1BA = b$ (Fig. 8) have radical axis s_a, while the triplet of circles obtained by the equalities $\mathbf{S}A_1BC = \mathbf{S}A_1CB = b$, $\mathbf{S}B_1CA = \mathbf{S}B_1AC = g$ and $\mathbf{S}C_1AB = \mathbf{S}C_1BA = a$ (Fig 9) has radical axis s_b. Analogously to proposition 2 we have

Proposition 4. *If* $s_b \cap s_c = A_3$, $s_c \cap s_a = B_3$ *and* $s_a \cap s_b = C_3$ *the triangle* $A_3B_3C_3$ *is equilateral and its center coinsides with the center of* ΔABC (Fig. 10).

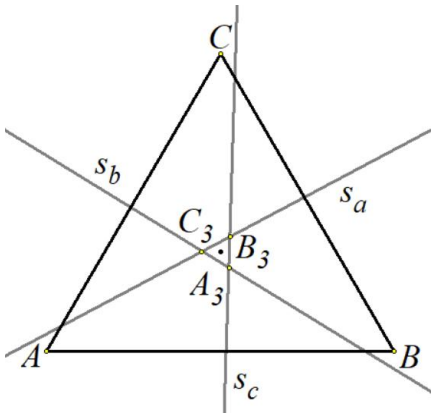

Fig. 10

5. Ellipses with common points. Ellipses could be circumscribed concerning the triangles AA_1A_2, BB_1B_2 and CC_1C_2, considered in propositions 1 and 3. Their centers are the gravity centers of the corresponding triangles. Experiments by GSP show, that the following assirtions are possible:

Proposition 5. *Given is an equilateral triangle ABC. The points* A_1, B_1 *and* C_1 *are interior for the triangle satisfying* $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$ *and* $\mathbf{S}BA_1C + \mathbf{S}CB_1A + \mathbf{S}AC_1B = 480°$. *Let* BC_1 *and* CB_1 *intersect in* A_2, CA_1 *and* AC_1 *intersect in* B_2, *while* AB_1 *and* BA_1 *intersect in* C_2. *The circumscribed ellipses of the triangles* AA_1A_2, BB_1B_2 *and* CC_1C_2, *whose centers are the gravity centers of the corresponding triangles, have two common points* (Fig. 11).

Proposition 6. *Given is an equilateral triangle ABC. The points* A_1 *and* B_1 *are interior for the triangle, while* C_1 *is exterior and* $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, $\mathbf{S}BA_1C + \mathbf{S}CB_1A - \mathbf{S}AC_1B = 120°$. *Let* BC_1 *and* CB_1 *intersect in* A_2, CA_1 *and* AC_1 *intersect in* B_2, *while* AB_1 *and* BA_1 *intersect in* C_2. *The circumscribed ellipses of the triangles* AA_1A_2, BB_1B_2 *and* CC_1C_2, *whose centers are the gravity centers of the corresponding triangles, have two common points* (Fig. 12).

When trying to prove these assertions, the general results obtained in the process of proving proposition 1 could be applied. Strict results are not available for the moment. For this reason propositions 5 and 6 are still hypotheses only.

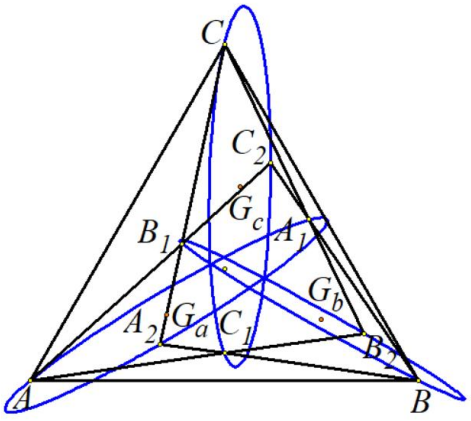

Fig. 11

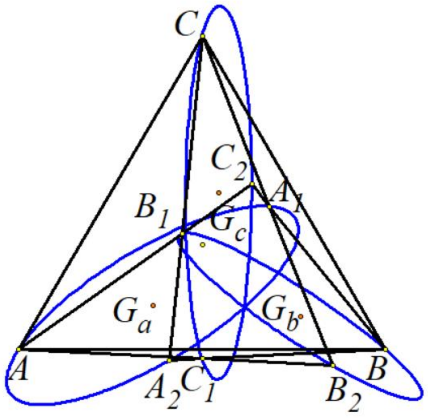

Fig. 12

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