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The Power Center of Three Circles in a Cyclic Quadrilateral is the Diagonal Intersection Point

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1. Introduction

There are a handful of famous lines in a given quadrilateral. The most famous is arguably the Newton line (the line connecting the midpoints of the diagonals), for which Newton's theorem states that in a tangential quadrilateral, the center of the incircle lies on the Newton line.
Here, we prove a result about another line in a quadrilateral. Specifically we prove that for a cyclic quadrilateral, the power center of the circumcircle and the two circles passing through opposite pairs of midpoints and the circumcenter is the point of intersection of the diagonals.

2. Results

Theorem 2.1. *In a quadrilateral, the intersection point of the diagonals lies on the radical line of the two circles passing through opposite midpoints and the points that complete the quadrilateral.*

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Proof. Let A, B, C, and D be the vertices of a given cyclic quadrilateral. Without loss of generality, let the cartesian coordinates of A, B, C, and D be respectively given by $(0,0), (1,0), (x_c, y_c), (x_d, y_d)$.
Let E have coordinates given by $(\frac{1}{2}, 0)$.
Let F have coordinates given by $(\frac{x_c+1}{2}, \frac{y_c}{2})$.
Let G have coordinates given by $(\frac{x_c+x_d}{2}, \frac{y_c+y_d}{2})$.
Let H have coordinates given by $(\frac{x_d}{2}, \frac{y_d}{2})$.
Line AC is given by the equation $y = (\frac{y_c}{x_c})x$.
Line BD is given by the equation $y = (\frac{y_d}{x_d-1})(x-1)$.
Therefore, the point of intersection of the diagonals, M, has coordinates given by $(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}, (\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))$.
Let lines AD and BC intersect at I and lines AB and CD intersect at J.
Line AD is given by the equation $y = (\frac{y_d}{x_d})x$.
Line BC is given by the equation $y = (\frac{y_c}{x_c-1})(x-1)$.
Therefore, the coordinates of I are given by $(\frac{y_c}{y_c-(x_c-1)(\frac{y_d}{x_d})}, (\frac{y_d}{x_d})(\frac{y_c}{y_c-(x_c-1)(\frac{y_d}{x_d})}))$.
Line AB is given by the equation $y = 0$.
Line CD is given by the equation $y - y_c = (\frac{y_c-y_d}{x_c-x_d})(x-x_c)$.
Therefore, the coordinates of J are given by $(x_c - \frac{y_c(x_c-x_d)}{y_c-y_d}, 0)$.
The circle passing through E, G, and J is given by the equation
$(x - \frac{y_d(2x_c+1)-y_c(2x_d+1)}{4(y_d-y_c)})^2 + (y - \frac{y_d^2-y_c^2-x_c^2+x_c+x_d^2-x_d}{4(y_d-y_c)})^2 =$
$(\frac{y_d(2x_c+1)-y_c(2x_d+1)}{4(y_d-y_c)} - \frac{1}{2})^2 + (\frac{y_d^2-y_c^2-x_c^2+x_c+x_d^2-x_d}{4(y_d-y_c)})^2$.
The circle passing through F, H, and I is given by the equation
$(x + \frac{y_d(y_c(y_d-y_c)-x_c^2+1)+x_dy_c(x_d+2)}{4(y_d(x_c-1)-x_dy_c)})^2 + (y - \frac{y_d^2(x_c-1)-2y_cy_d-x_dy_c^2+x_d^2(x_c-1)+x_d(1-x_c^2)}{4(y_d(x_c-1)-x_dy_c)})^2 =$
$(\frac{1}{2}x_d + \frac{y_d(y_c(y_d-y_c)-x_c^2+1)+x_dy_c(x_d+2)}{4(y_d(x_c-1)-x_dy_c)})^2 + (\frac{1}{2}y_d - \frac{y_d^2(x_c-1)-2y_cy_d-x_dy_c^2+x_d^2(x_c-1)+x_d(1-x_c^2)}{4(y_d(x_c-1)-x_dy_c)})^2$.
Therefore the radical line between the two circles is given by the equation
$(y_cy_d^3 - (2y_c^2 - (x_c-1)x_c)y_d^2 + y_c(y_c^2 + x_d(x_d - 4x_c + 3) + (x_c-1)x_c)y_d + (x_d - 1)x_dy_c^2)x - y_cy_d^3 + (y_c^2 - (x_d - x_c - 1)yy_c - x_c^2 + x_c)y_d^2 + ((x_d - x_c - 1)yy_c^2 - x_d(x_d - 2x_c + 1)y_c + (x_c-1)x_c(x_d - x_c + 1)y)y_d - (x_d-1)x_d(x_d - x_c + 1)yy_c = 0$.
We then verify that M lies on this line since
$(y_cy_d^3 - (2y_c^2 - (x_c-1)x_c)y_d^2 + y_c(y_c^2 + x_d(x_d - 4x_c + 3) + (x_c-1)x_c)y_d + (x_d - 1)x_dy_c^2)x_M - y_cy_d^3 + (y_c^2 - (x_d - x_c - 1)y_My_c - x_c^2 + x_c)y_d^2 + ((x_d - x_c - 1)y_My_c^2 - x_d(x_d - 2x_c + 1)y_c + (x_c-1)x_c(x_d - x_c + 1)y_M)y_d - (x_d-1)x_d(x_d - x_c + 1)y_My_c$
$= (y_cy_d^3 - (2y_c^2 - (x_c-1)x_c)y_d^2 + y_c(y_c^2 + x_d(x_d - 4x_c + 3) + (x_c-1)x_c)y_d + (x_d - 1)x_dy_c^2)(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}) - y_cy_d^3 + (y_c^2 - (x_d - x_c - 1)((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))y_c - x_c^2 + x_c)y_d^2 + ((x_d - x_c - 1)((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))y_c^2 - x_d(x_d - 2x_c + 1)y_c + (x_c-1)x_c(x_d - x_c + 1)((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)})))y_d - (x_d-1)x_d(x_d - x_c + 1)((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))y_c$
$= 0$ as desired. □

Theorem 2.2. *In a cyclic quadrilateral, the power center of the circumcircle and the two circles passing through opposite midpoints and the circumcenter is the point of intersection of the diagonals.*

Proof. Let A, B, C, and D be the vertices of a given cyclic quadrilateral. Without loss of generality, let the cartesian coordinates of A, B, C, and D be respectively given by $(0,0), (1,0), (x_c, y_c), (x_d, y_d)$.
Let E have coordinates given by $(\frac{1}{2}, 0)$.

Let F have coordinates given by $(\frac{x_c+1}{2}, \frac{y_c}{2})$.
Let G have coordinates given by $(\frac{x_c+x_d}{2}, \frac{y_c+y_d}{2})$.
Let H have coordinates given by $(\frac{x_d}{2}, \frac{y_d}{2})$.
Line AC is given by the equation $y = (\frac{y_c}{x_c})x$.
Line BD is given by the equation $y = (\frac{y_d}{x_d-1})(x-1)$.
Therefore, the point of intersection of the diagonals, O, has coordinates given by $(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}, (\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))$.
The circumcircle is given by the equation $(x-\frac{1}{2})^2+(y+\frac{x_c-x_c^2-y_c^2}{2y_c})^2 = \frac{y_c^2+(x_c-x_c^2-y_c^2)^2}{4y_c^2}$
$\iff (x-\frac{1}{2})^2 + y^2 + (\frac{y(x_c-x_c^2-y_c^2)}{y_c}) + \frac{(x_c-x_c^2-y_c^2)^2}{4y_c^2} = \frac{1}{4} + \frac{(x_c-x_c^2-y_c^2)^2}{4y_c^2}$
$\iff (x-\frac{1}{2})^2 + y^2 + \frac{y(x_c-x_c^2-y_c^2)}{y_c} = \frac{1}{4}$.
Since the quadrilateral is cyclic we know that x_d and y_d satisfy
$(x_d-\frac{1}{2})^2 + y_d^2 + \frac{y_d(x_c-x_c^2-y_c^2)}{y_c} = \frac{1}{4}$.
The coordinates of I are given by $(\frac{1}{2}, \frac{x_c^2+y_c^2-x_c}{2y_c})$.
The circle passing through E, G, and I is given by the equation
$(x - \frac{y_d(2x_c+1)-y_c(2x_d+1)}{4(y_d-y_c)})^2 + (y - \frac{y_d^2-y_c^2-x_c^2+x_c+x_d^2-x_d}{4(y_d-y_c)})^2 =$
$(\frac{y_d(2x_c+1)-y_c(2x_d+1)}{4(y_d-y_c)} - \frac{1}{2})^2 + (\frac{y_d^2-y_c^2-x_c^2+x_c+x_d^2-x_d}{4(y_d-y_c)})^2$.
The circle passing through F, H, and I is given by the equation
$(x + \frac{y_d(y_c(y_d-y_c)-x_c^2+1)+x_dy_c(x_d+2)}{4(y_d(x_c-1)-x_dy_c)})^2 + (y - \frac{y_d^2(x_c-1)-2y_cy_d-x_dy_c^2+x_d^2(x_c-1)+x_d(1-x_c^2)}{4(y_d(x_c-1)-x_dy_c)})^2 =$
$(\frac{1}{2}x_d + \frac{y_d(y_c(y_d-y_c)-x_c^2+1)+x_dy_c(x_d+2)}{4(y_d(x_c-1)-x_dy_c)})^2 + (\frac{1}{2}y_d - \frac{y_d^2(x_c-1)-2y_cy_d-x_dy_c^2+x_d^2(x_c-1)+x_d(1-x_c^2)}{4(y_d(x_c-1)-x_dy_c)})^2$.
Therefore, the radical line between the circumcircle and the circle passing through E, G, and I is given by the equation
$y_c((2x_c-1)y_d - (2x_d-1)y_c)x + yy_cy_d^2 - (2yy_c^2 + x_cy_c + 2x_c^2y - 2x_cy)y_d + yy_c^3 + x_dy_c^2 + ((x_d-1)x_d + (x_c-1)x_c)yy_c = 0$.
We just need to verify that O lies on this line, since the radical line between the two circles passing through opposite midpoints and the points that complete the quadrilateral passes through O by the previous theorem, and since the third radical line always passes through the intersection point of the other two. Now using the fact that $x_d = \frac{1}{2} \pm \sqrt{\frac{1}{4} - y_d^2 - \frac{y_d(x_c-x_c^2-y_c^2)}{y_c}}$ we then verify that O lies on this line since
$y_c((2x_c-1)y_d - (2x_d-1)y_c)x_O + y_Oy_cy_d^2 - (2y_Oy_c^2 + x_cy_c + 2x_c^2y_O - 2x_cy_O)y_d + y_Oy_c^3 + x_dy_c^2 + ((x_d-1)x_d + (x_c-1)x_c)y_Oy_c$
$= y_c((2x_c-1)y_d - (2x_d-1)y_c)(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}) + ((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))y_cy_d^2$
$- (2((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))y_c^2 + x_cy_c + 2x_c^2((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))$
$- 2x_c((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)})))y_d + ((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))y_c^3 + x_dy_c^2$
$+ ((x_d-1)x_d + (x_c-1)x_c)((\frac{y_c}{x_c})(\frac{y_d}{y_d-(\frac{y_c}{x_c})(x_d-1)}))y_c$
$= \frac{y_c(y_d-y_c)(y_d(y_c(y_d-y_c)-x_c(x_c-1))+x_dy_c(x_d-1))}{x_cy_d-y_c(x_d-1)}$
$= \frac{y_c(y_d-y_c)(y_d(y_c(y_d-y_c)-x_c(x_c-1))+(\frac{1}{2}+\sqrt{\frac{1}{4}-y_d^2-\frac{y_d(x_c-x_c^2-y_c^2)}{y_c}})y_c((\frac{1}{2}+\sqrt{\frac{1}{4}-y_d^2-\frac{y_d(x_c-x_c^2-y_c^2)}{y_c}})-1))}{x_cy_d-y_c((\frac{1}{2}+\sqrt{\frac{1}{4}-y_d^2-\frac{y_d(x_c-x_c^2-y_c^2)}{y_c}})-1)}$
$= \frac{y_c(y_d-y_c)(y_d(y_c(y_d-y_c)-x_c(x_c-1))+(\frac{1}{2}-\sqrt{\frac{1}{4}-y_d^2-\frac{y_d(x_c-x_c^2-y_c^2)}{y_c}})y_c((\frac{1}{2}-\sqrt{\frac{1}{4}-y_d^2-\frac{y_d(x_c-x_c^2-y_c^2)}{y_c}})-1))}{x_cy_d-y_c((\frac{1}{2}-\sqrt{\frac{1}{4}-y_d^2-\frac{y_d(x_c-x_c^2-y_c^2)}{y_c}})-1)}$
$= 0$ as desired. □

It follows from this and from [2] that in a given cyclic quadrilateral, the line bisecting the angle formed by the Miquel Steiner point and the points that complete the quadrilateral is the same line as the radical line of the two circles passing through opposite midpoints and the points that complete the quadrilateral.

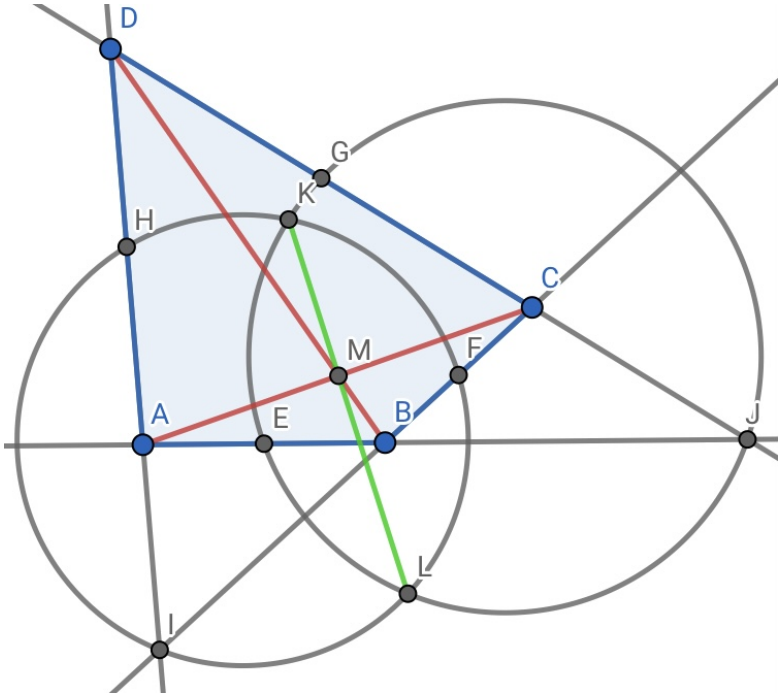

Figure 1. The intersection point of the diagonals lies on the radical line of the two circles passing through opposite midpoints and the points that complete the quadrilateral

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[2] Warren, B.L. "The Line that Bisects the Angle Formed by the Miquel Steiner point and the Two Points that Complete a Quadrilateral." To appear in the International Journal of Computer Discovered Mathematics.

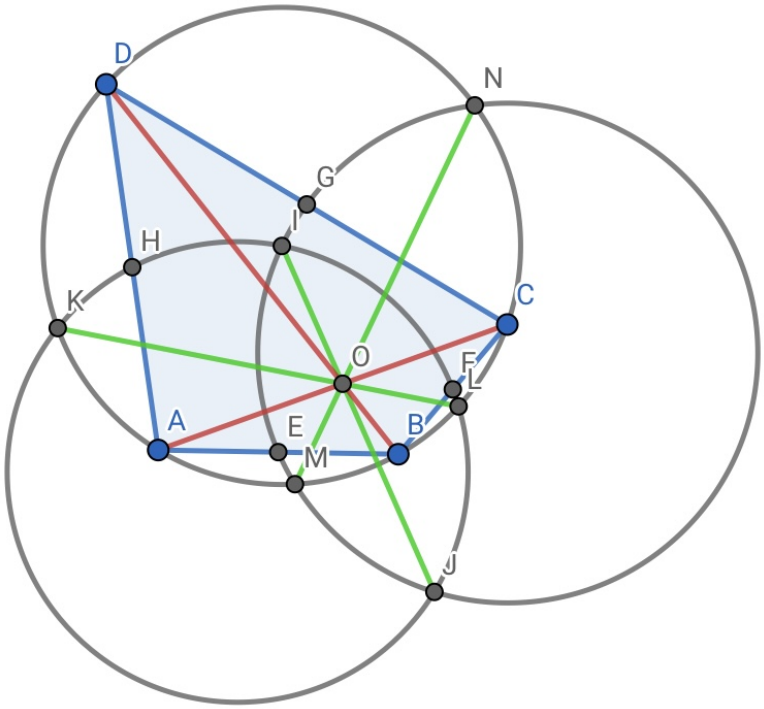

Figure 2. The power center O of the circumcircle and the circles passing through opposite midpoints and the circumcenter is the point of intersection of the diagonals.