International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 8, 2023, pp. 85-90.
Received December 5, 2023. Published on-line February 14, 2024.
web: http://www.journal-1.eu/
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Generalization of a Triangle Type Criterion

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Abstract. We derive a result about a direct correspondence between an angle of a given triangle and the ratio between the distance from a vertex to an excenter and the distance from that vertex to the incenter.

Keywords. incenter, excenter.

Mathematics Subject Classification (2010). 51N20.

1. Introduction

In [1], the following result was proven.

Theorem 1.1. *Let A, B, and C be the vertices of any triangle.*
1. ΔABC is acute $\iff$ two of the excenters of ΔABC each lie further from two vertices than the incenter does to either of those vertices.
2. ΔABC is a right triangle $\iff$ two of the excenters each lie the same distance from two vertices as the incenter does to either of those vertices.
3. ΔABC is obtuse $\iff$ two of the excenters of ΔABC each lie closer to two vertices than the incenter does to either of those vertices.

One can ask a stronger question: given the ratio of distances from a vertex to an excenter and the distance from that vertex to the incenter, is there a way to determine what the angle in question is? We see from the above that if we let R be this ratio, then $R < 1$ indicates that angle to be greater than $\frac{\pi}{2}$, if $R = 1$ then that angle is $\frac{\pi}{2}$, and if $R > 1$ then that angle is less than $\frac{\pi}{2}$.
Here we strengthen this result and provide the conclusive answer as to what that angle is given the precise ratio.

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2. Results

Theorem 2.1. *Let A, B, and C denote the vertices of a given triangle.*
Let D denote the incenter of ΔABC.
Let E denote the excenter on the line CD, F denote the excenter on the line AD, and G denote the excenter on the line BD.
Let α denote the distance from A to E.
Let β denote the distance from A to D.
Let γ denote the distance from C to F.
Let δ denote the distance from C to D.
Then we have the following results.
1. $R = \frac{\alpha}{\beta} = \frac{\gamma}{\delta}$.
2. $\angle ABC = \pi - 2\arctan(R)$.

Proof 1. Without loss of generality, we take the cartesian coordinates of A, B, and C to be given respectively by $(0,0)$, $(0,x_B)$, and (x_C,y_C).
D has coordinates given by
$(\frac{x_B\sqrt{x_C^2+y_C^2}+x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B}, \frac{x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})$.
E has coordinates given by
$(\frac{x_B\sqrt{x_C^2+y_C^2}-x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B}, \frac{-x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B})$.
Using the fact that the incenter of a triangle is the orthocenter of the excentral triangle, we see that since $\angle ADE = \angle CDF$, and $\angle DAE$ and $\angle DCF$ are right angles, then ΔADE and ΔCDF are similar triangles. Thus we have that $R = \frac{\alpha}{\beta} = \frac{\gamma}{\delta}$.
So now we calculate

$$R^2 = \frac{\alpha^2}{\beta^2} = \frac{(\frac{x_B\sqrt{x_C^2+y_C^2}-x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B})^2+(\frac{-x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}-x_B})^2}{(\frac{x_B\sqrt{x_C^2+y_C^2}+x_Bx_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2+(\frac{x_By_C}{\sqrt{(x_B-x_C)^2+y_C^2}+\sqrt{x_C^2+y_C^2}+x_B})^2}$$

$$= 1 - \frac{2(x_C-x_B)(\sqrt{y_C^2+(x_C-x_B)^2}-x_C+x_B)}{y_C^2}$$

$$\iff y_C^2+(x_C-x_B)^2 = (\frac{y_C^2(1-R^2)}{2(x_C-x_B)}+x_C-x_B)^2$$

$$\iff y_C^2+(x_C-x_B)^2 = (\frac{y_C^2(1-R^2)}{2(x_C-x_B)})^2+(x_C-x_B)^2+2(\frac{y_C^2(1-R^2)}{2(x_C-x_B)})(x_C-x_B)$$

$$\iff 1 = \frac{y_C^2(1-R^2)^2}{4(x_C-x_B)^2}+1-R^2$$

$$\iff R^2 = \frac{y_C^2(1-R^2)^2}{4(x_C-x_B)^2}$$

$$\iff \frac{4R^2}{(1-R^2)^2} = \frac{y_C^2}{(x_C-x_B)^2} = \tan^2(\phi) \text{ where } \phi = \pi - \angle ABC.$$

By carefully examining where we can take roots to have our scenario hold, we arrive at the following.

$$\angle ABC = \begin{cases} \pi - \arctan(\frac{2R}{1-R^2}) & R<1 \\ \frac{\pi}{2} & R=1 \\ \arctan(\frac{2R}{R^2-1}) & R>1. \end{cases}$$

Now, if we let $R = \tan(x)$ for non-negative x less than $\frac{\pi}{2}$, then we see that $\frac{2R}{1-R^2} = \frac{2\tan(x)}{1-\tan^2(x)} = \tan(2x)$ which leads to the above being reformulated as $\angle ABC = \pi - 2\arctan(R)$ for non-negative R. □

Proof 2. It will be fruitful to consider a few extra phantom figures.
Let the orthogonal projection of F onto the extended line of AB intersect this line at point H.
Let line AB intersect line CD at point I.
Consider the angle ϕ given by $\phi = \angle FBH = \arctan(\frac{FH}{BH})$.
Then clearly $\angle ABC = \pi - 2\phi$.
Since lines AB and EF intersect we have $\angle FBH = \angle ABE$.
Because $AD \perp AE$ and $BD \perp BE$, it follows that the quadrilateral $ADBE$ is a right kite, and therefore has a circumcircle.
Since AB and DE form two chords intersecting in the circumcircle of $ADBE$, it follows by the intersecting chords theorem [2] that ΔADI and ΔBEI are similar triangles. In particular, we have that $\angle EBI = \angle ADI$.
Now, since line DI intersects line AD, we see that $\angle ADI = \angle CDF$, and since $\angle DCF$ is a right angle, it follows that ΔCDF and ΔBFH are similar triangles.
So we have that $R = \frac{CF}{CD} = \frac{FH}{BH}$ and thus we conclude that indeed we have $\angle ABC = \pi - 2\arctan(R)$. □

3. An Application

An application of theorem 2.1 is as follows, and gives a rather clean result.

Theorem 3.1. *For a given angle in a triangle, let R denote the ratio of distances from an adjacent vertex to the excenter formed by the angle and the vertex and the distance from that vertex to the incenter.*
Let R_1, R_2, and R_3 denote these values for each of the angles. Then

$$R_1 + R_2 + R_3 = R_1R_2R_3. \tag{1}$$

Proof. Let A, B, and C denote the vertices of a given triangle. On the one hand, we know that $\angle ABC + \angle ACB + \angle BAC = \pi$.
But by theorem 2.1, it follows that
$\angle ABC + \angle ACB + \angle BAC$
$= (\pi - 2\arctan(R_1)) + (\pi - 2\arctan(R_2)) + (\pi - 2\arctan(R_3))$
$= 3\pi - 2(\arctan(R_1) + \arctan(R_2) + \arctan(R_3))$.
So putting these together gives
$3\pi - 2(\arctan(R_1) + \arctan(R_2) + \arctan(R_3)) = \pi$
$\iff \arctan(R_1) + \arctan(R_2) + \arctan(R_3) = \pi$.
Now the identity $\arctan(a) + \arctan(b) = \arctan(\frac{a+b}{1-ab})$ gives
$\arctan(R_1) + \arctan(R_2) + \arctan(R_3) = \arctan(\frac{R_1 + \frac{R_2+R_3}{1-R_2R_3}}{1-R_1(\frac{R_2+R_3}{1-R_2R_3})}) = \pi$.
Taking tangents on both sides gives
$\tan(\arctan(\frac{R_1 + \frac{R_2+R_3}{1-R_2R_3}}{1-R_1(\frac{R_2+R_3}{1-R_2R_3})})) = \tan(\pi) \implies \frac{R_1 + \frac{R_2+R_3}{1-R_2R_3}}{1-R_1(\frac{R_2+R_3}{1-R_2R_3})} = 0$
$\implies R_1 + \frac{R_2+R_3}{1-R_2R_3} = 0 \implies R_1 + R_2 + R_3 = R_1R_2R_3.$ □

Thus it directly follows that if R_1 and R_2 are rational where $R_1R_2 = 1$, then R_3 is rational.

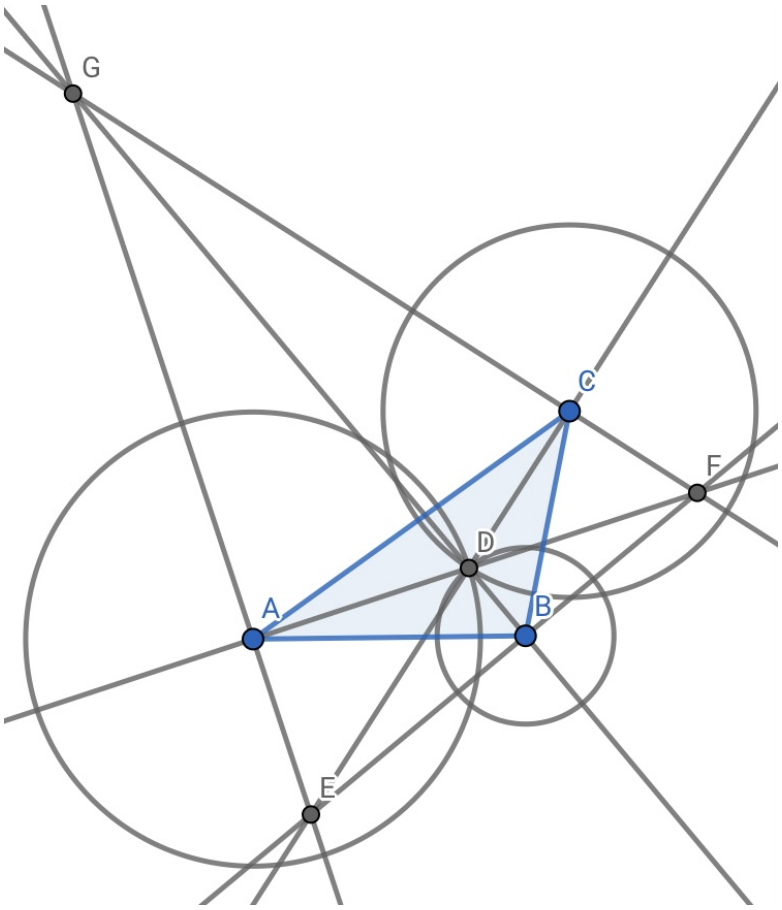

FIGURE 1.
$\angle ABC = \pi - 2\arctan(\frac{\text{distance from A to E}}{\text{distance from A to D}})$
$= \pi - 2\arctan(\frac{\text{distance from C to F}}{\text{distance from C to D}})$.

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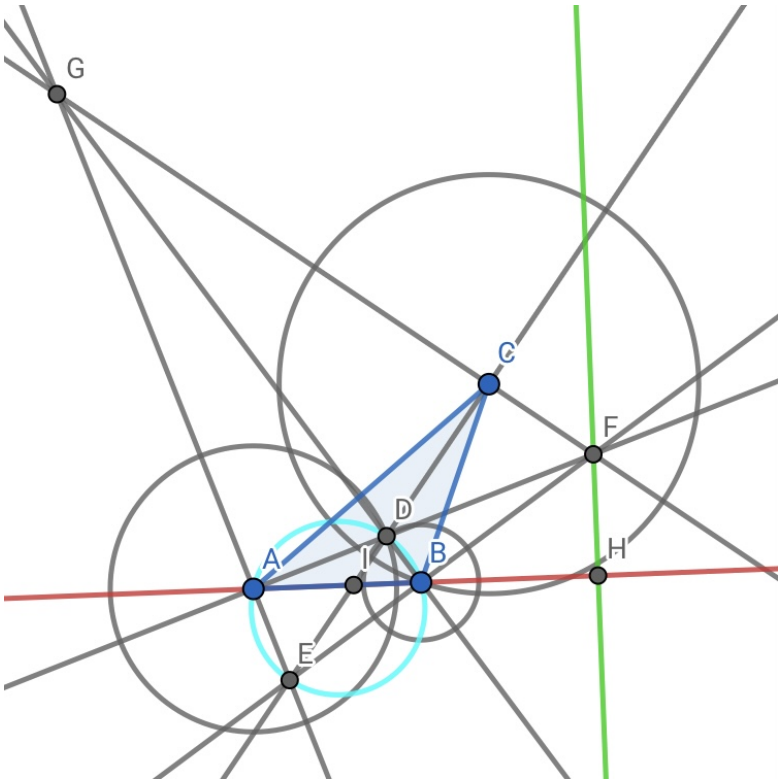

Figure 2. Extra phantom figures, showing that $\angle FBH = \angle CDF$.

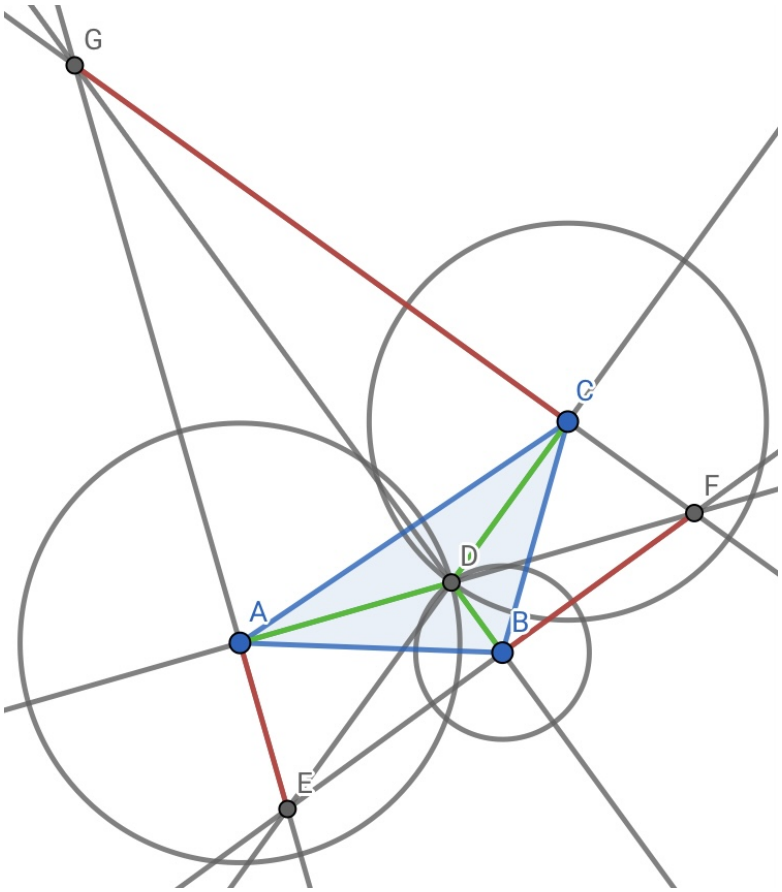

FIGURE 3. $R_1 = \frac{AE}{AD} = \frac{CF}{CD}$, $R_2 = \frac{BF}{BD} = \frac{AG}{AD}$, $R_3 = \frac{CG}{CD} = \frac{BE}{BD}$. $R_1 + R_2 + R_3 = R_1 R_2 R_3$.