A Computer Algorithm for Proving Symmetric Homogeneous Triangle Inequalities

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Abstract. We present an algorithm for efficiently proving symmetric homogeneous polynomial triangle inequalities using Blundon's Fundamental Inequality. If the inequality contains an undetermined constant k, the algorithm automatically finds the best value of k for which the inequality is true.

Keywords. triangle geometry, inequalities, computer-discovered mathematics, Mathematica, fundamental region.

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1. Introduction

Let the sides of a triangle be a, b, and c, and let r and R be the inradius and circumradius, respectively. Let K denote the area of the triangle and let s denote its semiperimeter.

There is a vast amount of literature concerning inequalities between these quantities. A typical inequality is

$$a^2+b^2+c^2 \le 8R^2+4r^2 \tag{1}$$

which is inequality 5.16 in [4]. Compendiums of such inequalities can be found in [4], [20], and [10].

It is the purpose of this paper to present an efficient algorithm for proving such inequalities. The algorithm handles symmetric homogeneous inequalities involving rational functions of the quantities a, b, c, R, r, s, and K, as well as other elements of a triangle, such as the exradii and trigonometric functions of the angles of the triangle.

The algorithm can also discover new inequalities if a suspected inequality is of a form containing an undetermined constant k, such as

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{1}{\sqrt{3}}\left[\frac{1}{R}+\frac{1}{r}+\frac{1}{k}\left(\frac{2}{R}-\frac{1}{r}\right)\right]$$

which was proposed as a problem by Huang in [17] and discussed in [39]. Our algorithm found that the best constant k that makes this inequality true is

$$k=2\left(1+\sqrt[3]{2}+\sqrt[3]{4}\right)$$

which agrees with the value found by Chen in [9].

2. Background Information

The types of inequalities we are interested in fall into the study of *real closed fields*. An inequality such as $x^2+y^2+z^2\ge xy+yz+zx$ can be formally represented in such a field by the formula

$$(\forall x)(\forall y)(\forall z)\left((x\ge 0)\wedge(y\ge 0)\wedge(z\ge 0)\Rightarrow x^2+y^2+z^2\ge xy+yz+zx\right).$$

The symbols such as $\exists$ and $\forall$ are known as *quantifiers*. In 1930, Tarski discovered a decision procedure for the formal theory of real closed fields. His method was first published in 1948 [29]. Tarski's method involves successive elimination of quantifiers using Sturm sequences. This method can be used to automatically prove all the inequalities discussed in this paper as well as many problems in Euclidean geometry. Unfortunately, Tarski's algorithm, although totally effective, is completely impractical given the state of the art of computers today. The same holds true for an improvement given by Seidenberg, [26].

Over the years, improved methods have been devised for effectively proving results in the theory of real closed fields. For example, Collins [13] devised a method that employs cylindrical decomposition using Gröbner bases to eliminate quantifiers. See Davenport [14], section 3.2, for an exposition. See also [12].

The state of the art has improved so much that modern computer algebra systems, such as Mathematica, can now automatically solve large classes of systems of equations and inequalities. According to the Mathematica technical documentation, their function `Solve` uses cylindrical algebraic decomposition and Gröbner basis methods [38] using an efficient version of the *Buchberger Algorithm* [7]. `Solve` can always, in principle, solve any system of polynomials equations and inequalities over the real domain [37].

For example, let us see how we can use Mathematica to prove inequality (1). Using the definitions $s=(a+b+c)/2$, $K=\sqrt{s(s-a)(s-b)(s-c)}$, $R=abc/(4K)$, and $r=K/s$, we can issue the following Mathematica commands.

```
s = (a+b+c)/2;
K = Sqrt[s(s-a)(s-b)(s-c)];
R = a*b*c/(4K);
r = K/s;
inequality = a^2+b^2+c^2 <= 8R^2+4r^2;
triang = a>0 && b>0 && c>0 && a+b>c && b+c>a && c+a>b;
Simplify[inequality, triang]
```

The final `Simplify` command tells Mathematica to simplify the specified inequality, subject to the specified restrictions that the variables a, b, and c are positive real numbers that satisfy the triangle inequality. In under a tenth of a second, Mathematica returns the response

```
True
```

indicating that the inequality is true.

As the complexity of the inequality increases, the running time becomes longer. For example, consider the inequality

$$\frac{11\sqrt{3}}{5R+12r} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

found in [39]. It took version 12.0 of Mathematica 4.5 minutes to prove this inequality using the same procedure running on a 3.5 GHz iMac. As the complexity of the inequality increases, the running time gets even larger.

The method of cylindrical decomposition may be too slow for proving the types of inequalities we are interested in. That is because our triangle inequalities generally involve three quantifiers, and these methods attempt to eliminate one quantifier at a time. Each elimination step causes an expression explosion. In our case, the expressions involved are symmetric; so a method that removes one quantifier at a time is bound to be non-optimal.

Blundon and others ([3], [5], [21]) have attacked the problem by first expressing the proposed inequality in terms of R, r, and s. We use Blundon's algorithm to provide a more efficient algorithm for proving triangle inequalities than the algorithms built in to Mathematica.

3. Theoretical Basis for the Algorithm

In 1965, Blundon [3] showed that a triangle with circumradius R, inradius r, and semiperimeter s exists if and only if R, r, and s satisfy what is now known as *Blundon's Fundamental Inequality*:

$$s^2(18Rr-9r^2-s^2)^2 \leq (s^2-3r^2-12Rr)^3. \tag{2}$$

Since $r>0$, this is algebraically equivalent to the inequality

$$4R(R-2r)^3 \geq (s^2-2R^2-10Rr+r^2)^2. \tag{3}$$

Given a proposed homogeneous inequality involving R, r, and s, only the ratios of R, r, and s are of interest. Following a variation of Bottema [5], we can let

$x = r/R$ and $y = s/R$. Applying these substitutions to inequality (2), Blundon's Fundamental Inequality becomes

$$(x^2+y^2)^2 + 12x^3 - 20xy^2 + 48x^2 - 4y^2 + 64x \le 0. \tag{4}$$

The region determined by inequality (4) and $x > 0$ and $y > 0$ is called the *Fundamental Region*. The region in the xy-plane representing (x, y) values that satisfy a given inequality is called the *inequality region*.

The Basis for Blundon's algorithm. A triangle inequality, expressed in terms of x and y will be true if and only if every point in the fundamental region satisfies the given inequality.

Alternative Description. An inequality expressed in terms of x and y will be true if and only if the inequality region contains the fundamental region.

Each point in the xy-plane corresponds to an equivalence class of triples (R, r, s). Those pairs (x, y) that determine a triangle lie inside the region, $\mathcal{R}$, bounded by the y-axis and the hypocycloid whose parametric representation is given by

$$\begin{cases} x = \dfrac{4(t^2-1)}{(t^2+1)^2} \\ y = \dfrac{8t^3}{(t^2+1)^2} \end{cases} \tag{5}$$

with $t > 1$. Note that our t is the reciprocal of the t used in [3]. The region $\mathcal{R}$ has cusps at $(0, 0)$, $(0, 2)$, and $(1/2, 3\sqrt{3}/2)$. The region $\mathcal{R}$ is shown in Figure 1.

The points on the bounding hypocycloid correspond to isosceles triangles. The point at $(1/2, 3\sqrt{3}/2)$ corresponds to an equilateral triangle. The points on the y-axis from $(0, 0)$ to $(0, 2)$ correspond to degenerate triangles.

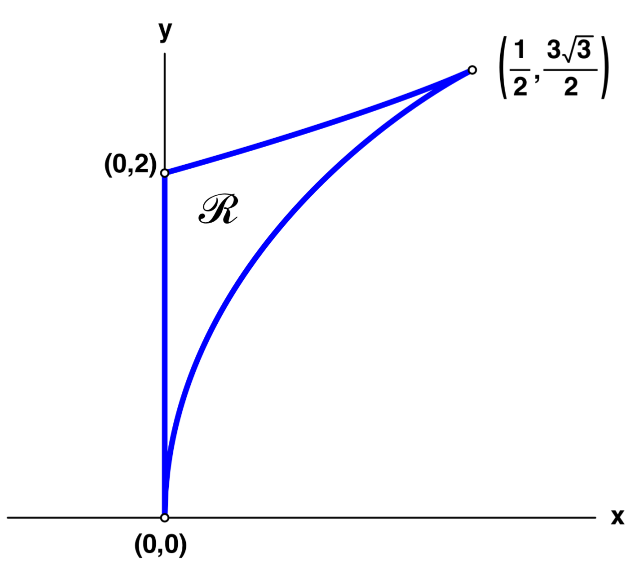


FIGURE 1. Fundamental Region

According to Bludon's original paper, [3], the geometrical significance of the parameter t is that t is the cotangent of one-half of the base angle of the isosceles triangle corresponding to t.

According to [20, p. 9], the equation for the upper boundary of the fundamental region is

$$y = \sqrt{2 + 10x - x^2 + 2(1-2x)^{3/2}}, \qquad 0 < x \le \frac{1}{2},$$

and the equation for the lower boundary is

$$y = \sqrt{2 + 10x - x^2 - 2(1-2x)^{3/2}}, \qquad 0 < x \le \frac{1}{2}.$$

4. Input to the Algorithm

The following symbols will be allowed in the inequalities to be proven. Their meaning is given in the following table.

Input Variables	
Notation	**Description**
a, b, c	The sides of the triangle
A, B, C	The angles of the triangle
K	The area of the triangle
s	The semiperimeter of the triangle
r	The inradius of the triangle
R	The circumradius of the triangle
$h(a), h(b), h(c)$	The altitudes of the triangle
$m(a), m(b), m(c)$	The medians of the triangle
$w(a), w(b), w(c)$	The angle bisectors of the triangle
$r(a), r(b), r(c)$	The exradii of the triangle
$g(a), g(b), g(c)$	The Gergonne cevians of the triangle
$n(a), n(b), n(c)$	The Nagel cevians of the triangle

5. Support Algorithms

Before describing our main algorithm, we present some algorithms that will be needed later.

Algorithm T [Remove Trigonometric Functions]

This algorithm removes trigonometric functions from an expression.

INPUT: Any expression containing the standard trigonometric functions of integer linear combinations of constants and the variables A, B, and C. Examples: $\cos^5 A + \sin(3B + 5C)$ and $\csc(2A + \pi/5)$. Also permitted are expressions of the form $\tan\frac{x}{2}$, $\sin^2\frac{x}{2}$, and $\cos^2\frac{x}{2}$ where x is any integer linear combination of constants and the variables A, B, and C.

STEP 1: [Remove Inverse Trigonometric Functions]
The trigonometric functions `sec`, `csc`, and `cot` are replaced by the reciprocals of the functions `cos`, `sin`, and `tan`, respectively.

STEP 2: [Remove Half Angles]
Allowable trigonometric functions of half angles are removed as follows where x denotes any integer linear combination of constants and the variables A, B, C.

$$\tan\frac{x}{2} \quad\rightarrow\quad \frac{\sin x}{1+\cos x}$$

$$\sin^2\frac{x}{2} \quad\rightarrow\quad \frac{1-\cos x}{2}$$

$$\cos^2\frac{x}{2} \quad\rightarrow\quad \frac{1+\cos x}{2}$$

STEP 3: [Remove Tangent]
The trigonometric function `tan x` is replaced by `(sin x)/(cos x)`.

STEP 4: [Remove Sums and Multiples]
The Mathematica function `TrigExpand` is used to get rid of trigonometric functions of multiple angles and sums of angles using familiar identities such as $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin(x+y) = \sin x\cos y + \cos x\sin y$.

STEP 5: [Evaluate constants]
Trigonometric functions of a constant are replaced by their numerical values. For example, $\sin(\pi/3)$ is replaced by $\sqrt{3}/2$.

STEP 6: [Convert to a-b-c Form]
The basic trigonometric functions are then replaced by expressions involving variables a, b, c, and R using standard formulas such as the *Law of Cosines* and the *Extended Sine Law* by applying substitutions such as the following.

$$\sin A \quad\rightarrow\quad \frac{a}{2R}$$

$$\cos A \quad\rightarrow\quad \frac{b^2+c^2-a^2}{2bc}$$

OUTPUT: The output of algorithm T is an equivalent expression that does not contain any trigonometric functions.

Algorithm R [Convert to R-r-s Form]

This algorithm takes a symmetric homogeneous expression (not involving trigonometric functions) and converts it to an equivalent expression involving only the symbols R, r, and s.

For this paper, we say that an expression $f(a,b,c)$ (not involving trigonometric functions) is *symmetric* if interchanging any two members from the set $\{a,b,c\}$ does not change the value of the expression. Note that by this definition, $ab^2+bc^2+ca^2$ is not a symmetric expression.

The expression $f(a,b,c)$ is said to be *homogeneous* if $f(at,bt,ct) = t^n f(a,b,c)$ for some constant n.

INPUT: Any symmetric homogeneous rational function of expressions involving the variables and functions listed in Section 4, except that only even powers of m,

w, g, and n are permitted. Trigonometric functions allowed as input to Algorithm T are also permitted.

STEP 1: [Remove Trigonometric Functions]

Apply Algorithm T to remove any trigonometric functions.

STEP 2: [Remove Triangle Elements]

Other triangle elements are removed using substitutions such as the following.

$$\begin{aligned}
K &\to rs \\
h(a) &\to \frac{2rs}{a} \\
r(a) &\to \frac{rs}{s-a} \\
m(a)^2 &\to \frac{1}{4}(2b^2+2c^2-a^2) \\
w(a)^2 &\to \frac{4bcs(s-a)}{(b+c)^2} \\
g(a)^2 &\to \frac{(s-a)(as-(b-c)^2)}{a} \\
n(a)^2 &\to \frac{c^2(s-b)+b^2(s-c)-a(s-b)(s-c)}{a}
\end{aligned}$$

These formulas are obtained by using Stewart's Theorem [2, p. 152].

STEP 3: [Remove a, b, and c]

The expression is now a homogeneous rational function of a, b, c, R, r, and s. Combine all terms giving an expression consisting of a numerator and a denominator. The numerator and denominator are symmetric functions of a, b, and c. By the *Fundamental Theorem of Symmetric Functions* [31], these expressions are converted into expressions involving R, r, and s by using the following transformations on the elementary symmetric polynomials.

$$\begin{cases} a+b+c \to 2s \\ ab+bc+ca \to r^2+s^2+4rR \\ abc \to 4rRs \end{cases} \tag{6}$$

These formulas come from [20, p. 7]. This transformation is performed by the Mathematica function `SymmetricReduction`.

OUTPUT: The output of algorithm R is an equivalent expression that contains only the variables R, r, and s.

Algorithm P [Remove Positive Factors]

This algorithm removes factors from a polynomial expression that are always positive subject to a given set of constraints. For example, subject to the constraint $x > 0$, this algorithm would remove factors like $7x + 5$, $x^2 + 1$, and $(x^3 - x + 17)^6$ from an expression because these factors are always positive when $x > 0$.

INPUT: The input to this algorithm is a polynomial expression, `expr`, in a set of variables, `vars`, and a set of constraints placed on those variables.

STEP 1: [Factor]

Factor `expr` as a product $\prod \mathrm{fact}_i^{e_i}$ where each fact_i is a squarefree polynomial. This can be done with the Mathematica function `FactorSquareFree`.

STEP 2: [Remove positive factors]

Examine each fact_i. Use the Mathematica command

```
Simplify[fact>0,constraints]
```

to determine if the expression fact_i is always positive subject to the given constraints. If so, remove the factor $\mathrm{fact}_i^{e_i}$.

OUTPUT The output of algorithm P is a polynomial which is positive if and only if the original expression was positive.

Technical note. Step 1 of this algorithm requires us to factor a polynomial, possibly involving multiple variables. Factoring multivariate polynomials is hard. According to [38], Mathematica factors a multivariate polynomial by substituting appropriate choices of integers for all but one variable, then factoring the resulting univariate polynomials and reconstructing multivariate factors using Wang's algorithm [30]. To factor a univariate polynomial, Mathematica uses a variant of the Cantor-Zassenhaus algorithm [8] to factor modulo a prime, then uses Hensel lifting and recombination [16] to build up factors over the integers.

Step 1 doesn't need to perform a full factorization. Instead, it is simpler to call the Mathematica function `FactorSquareFree` to find only factors of the form polynomial^n where $n > 1$. The function `FactorSquareFree` is faster than `Factor` because it works by finding a derivative and then iteratively computing GCDs [38].

Note 2. You might be tempted to improve this algorithm by removing any factors of the form fact^{2n} under the assumption that such expressions are positive. This would be incorrect. For example, the inequality $(x-2)^8(x^2-7x+1) \geq 0$ is not equivalent to the inequality $x^2-7x+1 \geq 0$ because the first inequality is true for $x = 2$ while the second one isn't.

Algorithm F [Remove Fractions from an Inequality]

This algorithm replaces an inequality involving rational functions of x and y with an equivalent inequality involving only polynomial functions of x and y.

INPUT: The input to this algorithm is an inequality of the form $\mathrm{expr} \geq 0$ or $\mathrm{expr} > 0$ where expr is a rational function of variables x and y satisfying the constraints $x > 0$ and $y > 0$.

STEP 1: [Gather Terms]

Gather all the terms together to put the inequality in the form

$$\frac{\mathrm{num}}{\mathrm{den}} \geq 0$$

where num and den are polynomials

STEP 2: [Remove Constants from Denominator]

If den is a positive constant, multiply both sides of the inequality by den. If den is a negative constant, multiply both sides of the inequality by $-\mathrm{den}$.

STEP 3: [Remove Positive Factors]

Apply Algorithm P with variables x and y using the constraints $x > 0$ and $y > 0$ to remove any positive factors occurring in either num or den. This does not change the truth value of the inequality because $fg \geq 0$ is equivalent to $f \geq 0$ if $g > 0$.

The inequality is now of the form

$$\frac{\text{num}}{\text{den}} \geq 0$$

where den is squarefree. Multiply both sides of the inequality by the positive quantity den^2. The inequality is now of the form

$$f(x, y) \geq 0,$$

where $f(x, y)$ is a polynomial.

OUTPUT: The output of algorithm F is an inequality that contains only polynomial functions of x and y and is equivalent to the given inequality.

6. The Algorithm

Algorithm B [Prove Triangle Inequality using Blundon's Method]

Algorithm B (short for Algorithm Blundon) takes a conjectured inequality and determines if the inequality is true or false.

INPUT: An inequality involving symmetric homogeneous rational functions of variables and functions listed in Section 4.

STEP 1: [Bring larger expression to left]

We change the inequality symbol to be $>$ or $\geq$ with the following substitutions.

$$\begin{aligned} x \leq y &\quad\rightarrow\quad y \geq x \\ x < y &\quad\rightarrow\quad y > x \end{aligned}$$

For the remainder of the description of this algorithm, we let the symbol $\geq$ denote either $>$ or $\geq$.

STEP 2: [Collect terms]

We bring all terms to the left side of the inequality using the following substitution.

$$x \geq y \quad\rightarrow\quad x - y \geq 0$$

STEP 3: [Convert to R-r-s Form]

Apply Algorithm R to the left side of the inequality to produce an equivalent inequality involving only the variables R, r, and s.

STEP 4: [Convert to x-y Form]

Combine all terms to form a single rational function on the left side of the inequality. This expression is now a fraction that contains only the variables R, r, and s. The numerator and denominator are homogeneous polynomials in R, r, and s, so only the ratios of R, r, and s are of interest.

To get rid of R, r, and s, we apply the following transformations.

$$\begin{aligned} r &\quad\rightarrow\quad xR \\ s &\quad\rightarrow\quad yR \end{aligned}$$

After simplifying the resulting expression, the inequality will now be of the form

$$f(x, y)R^n \geq 0.$$

Multiply both sides of the inequality by the positive quantity R^{-n} to get

$$f(x, y) \geq 0,$$

an inequality whose only variables are x and y. The function f is a rational function. It is not necessarily homogeneous.

STEP 5: [Remove fractions]

Apply Algorithm F to replace the inequality by an equivalent inequality in which the left side is a polynomial in x and y.

STEP 6: [Find the degree of $f(x, y)$]

The inequality is now of the form $f(x,y) \geq 0$, where f is a polynomial. Find the degree of the polynomial $f(x,y)$. This is a bit tricky in Mathematica because there is no built-in function to find the degree of a multivariate polynomial. The following function can be used.

```
polynomialDegree[poly_] :=
     Max[Plus@@@CoefficientRules[#][[All,1]]]&@poly;
```

Denote the degree by `deg`.

STEP 7: [Check the polynomial inequality $f(x,y) \geq 0$]

A triangle inequality, expressed in terms of x and y will be true if and only if every point in the fundamental region satisfies the given inequality.

If `deg` > 2, determine if the inequality $f(x,y) \geq 0$ is true by using the following Mathematica commands

```
fundamentalFunction = (x^2+y^2)^2+12x^3-20x*y^2+48x^2-4y^2+64x;
fundamentalRegion = fundamentalFunction<=0 && x>0 && y>0;
truth = Simplify[f[x,y] >= 0, fundamentalRegion];
```

If the simplified form of the inequality is `True`, then the inequality is true. If the simplified form of the inequality is not `True`, then the inequality is false. Set `truth` to `False`. Exit and return `truth` as the result of this algorithm.

If `deg` ≤ 2, a faster procedure can be used. Continue with step 8.

STEP 8: [Convert to t-form]

If `deg` ≤ 2, then the region of the inequality is convex. The triangle inequality, expressed in terms of x and y will be true if and only if every point on the boundary of the fundamental region satisfies the given inequality. (((need to check)))

We saw in Figure 1 that the boundary of the fundamental region is a portion of a hypocycloid and a portion of the y-axis. Points on the y-axis correspond to degenerate triangles, so it suffices to check that all the points on the hypocycloid portion satisfy the given inequality.

Use equation set (5) to express the boundary of the fundamental region as a function of t with $t > 1$. Make the following substitutions.

$$x \to \frac{4(t^2-1)}{(t^2+1)^2}$$
$$y \to \frac{8t^3}{(t^2+1)^2}$$

The given inequality is now equivalent to an inequality of the form $g(t) \geq 0$ subject to the constraint $t > 1$.

STEP 9: [Remove positive factors]

Apply Algorithm P with variable t using the constraint $t > 1$ to remove any positive factors occurring in either the numerator or denominator of g.

STEP 10: [Convert to polynomial]

Multiply both sides of the inequality by the square of the denominator to clear of fractions.

STEP 11: [Check the polynomial inequality $g(t) \geq 0$]

To see if the inequality $g(t) \geq 0$ is true, issue the following Mathematica command:

```
truth = Simplify[g[t] >= 0, t>1]
```

If the simplified form of the inequality is `True`, then the inequality is true. If the simplified form of the inequality is not `True`, then the inequality is false. Set `truth` to `False`. Return `truth` as the result of this algorithm.

OUTPUT: Algorithm B returns true or false indicating the correctness of the conjectured inequality.

7. Example

Let us illustrate the algorithm by working through the steps on the inequality

$$\frac{11\sqrt{3}}{5R+12r} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Step 1 turns this into the equivalent inequality

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{11\sqrt{3}}{5R+12r}.$$

Step 2 turns this into the equivalent inequality

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{11\sqrt{3}}{5R+12r} \geq 0.$$

For **Step 3**, we combine all terms to get the equivalent inequality

$$\frac{-11\sqrt{3}abc+12abr+5abR+12acr+5acR+12bcr+5bcR}{abc(12r+5R)} \geq 0.$$

Then we do a symmetric reduction, to get the equivalent inequality

$$\frac{-11\sqrt{3}(abc)+(12r+5R)(ab+bc+ca)}{(abc)(12r+5R)} \geq 0.$$

Now we replace the elementary symmetric functions by their equivalents using r, R, and s, using equation set (6), to get the equivalent inequality

$$\frac{-11\sqrt{3}(4rRs)+(12r+5R)(r^2+s^2+4rR)}{(4rRs)(12r+5R)} \geq 0.$$

Step 4 uses $r=xR$ and $s=yR$ to give the equivalent inequality

$$\frac{-11\sqrt{3}(4xyR^3)+(12xR+5R)((xR)^2+(yR)^2+4xR^2)}{(4xyR^3)(12xR+5R)} \geq 0.$$

Simplifying the expression on the left gives

$$\frac{-11\sqrt{3}(4xy)+(12x+5)(x^2+y^2+4x)}{(4xy)(12x+5)R} \geq 0.$$

Multiply both sides of the inequality by R and simplifying gives the equivalent inequality

$$\frac{12x^3+53x^2+12xy^2-44\sqrt{3}xy+20x+5y^2}{(4xy)(12x+5)} \geq 0.$$

In **step 5**, we note that the numerator does not factor and every factor in the denominator is positive because $x > 0$ and $y > 0$. We therefore multiply both sides of the inequality by the denominator to get the equivalent inequality

$$12x^3 + 53x^2 + 12xy^2 - 44\sqrt{3}xy + 20x + 5y^2 \geq 0.$$

Per **Step 6**, we note that the degree of this polynomial is 3.

For **Step 7**, we need to determine if every point in the Fundamental Region satisfies the given inequality. That is, we want to know if the inequality region contains the Fundamental Region.

Figure 2 (left) shows a portion of the graph of `func` ≥ 0 in the portion of the plane bounded by $-0.2 \leq x \leq 1.2$ and $-0.2 \leq y \leq 3$. The points in the region are yellow. This is the inequality region. Its boundary is red.

Figure 2 (right) shows the Fundamental Region in green. Its boundary is blue.

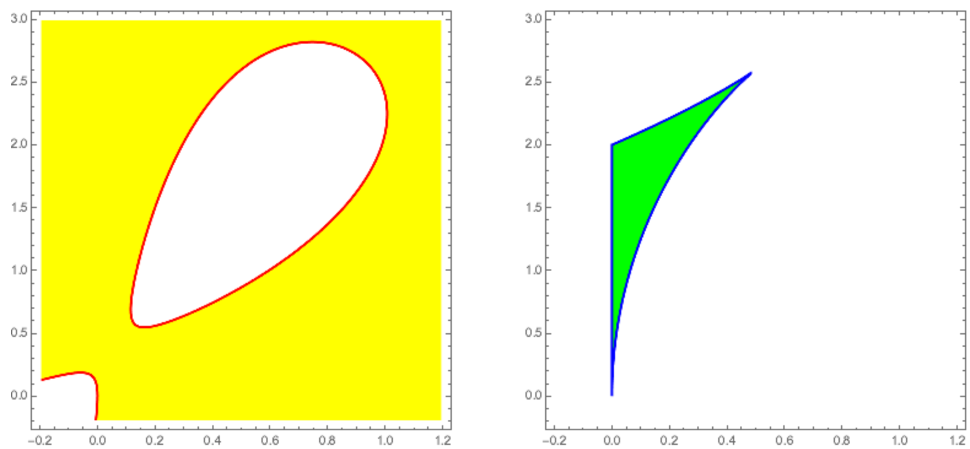


Figure 2. Inequality Region (yellow) and Fundamental Region (green)

Figure 3 shows both regions plotted on the same coordinate plane.

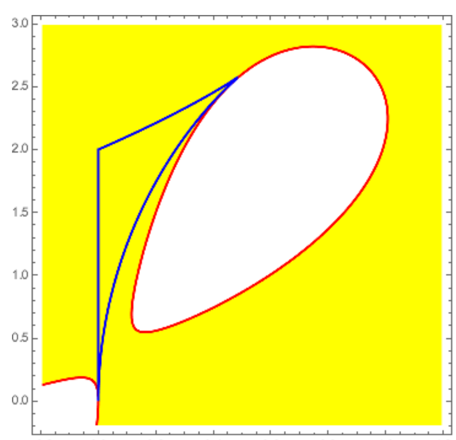


Figure 3. Inequality Region (yellow) and Fundamental Region (blue boundary)

The plots in Figure 2 seem to indicate that every point in the Fundamental Region (inside the blue curve) lies within the inequality region (yellow). This shows graphically that the inequality is true. The white region consists of those points that are not in either region.

However, examining the relationship of the regions by eye is not good enough. We use the procedure given in **Step 7** by issuing the following commands

```
fundamentalRegion = (x^2+y^2)^2+12x^3-20xy^2+48x^2-4y^2+64x<=0
                    && x>0 && y>0;
inequality = 12x^3+53x^2+12x*y^2-44Sqrt[3]x*y+20x+5y^2>=0;
Simplify[inequality, fundamentalRegion]
```

which determines if the inequality `inequality` is true subject to the constraint given by the the inequalities comprising `fundamentalRegion`.

The Mathematica output from the `Simplify` call is `True`. This means that the given inequality is true. The run time of the algorithm was 0.016 sec, compared to the 4.5 minutes needed if just using the Mathematica `Simplify` command.

8. Finding a counterexample

Algorithm C [Find Counterexample to Inequality]

If Algorithm B declares that an inequality is false, the following procedure can be used to exhibit a counterexample. Suppose that in step 11,

```
truth = Simplify[g[t] >= 0, t>1]
```

did not return `True`.

- Use the following Mathematica command to find a value of t that makes the inequality false.
  ```
  FindInstance[!(g[t]>=0) && t>1, t]
  ```
- Use that value of t to find the corresponding values of x and y.

$$\begin{cases} x = \dfrac{4(t^2-1)}{(t^2+1)^2} \\ y = \dfrac{8t^3}{(t^2+1)^2} \end{cases} \tag{7}$$

- Use these values of x and y to find r, R, and s. Without loss of generality, assume that $R = (t^2+1)^2$. Then

$$\begin{cases} R = (t^2+1)^2 \\ r = 4(t^2-1) \\ s = 8t^3 \end{cases} \tag{8}$$

- This gives you the counterexample in terms of r, R, and s. If you want to express the counterexample in terms of a, b, and c, use the fact that a, b, and c are the roots of the equation

$$x^3 - (2s)x^2 + (r^2+s^2+4rR)x - (4rRs) = 0. \tag{9}$$

This fact follows from Vieta's Formulas [35] and equation set (6).

If Algorithm B did not get to step 11 because `deg`> 2, we can construct a counterexample from the polynomial inequality found in step 7.

In this case, use the following Mathematica commands to find values for x and y that make the inequality false.

```
fundamentalRegion = fundamentalFunction<=0 && x>0 && y>0;
FindInstance[!(f[x,y] >= 0) && fundamentalRegion, {x,y}]
```

This gives you the counterexample in terms of x and y. Use the relationships

$$\begin{cases} r = xR \\ s = yR \end{cases} \tag{10}$$

to find r, R, and s. Only their ratios are important, so any convenient value for R can be chosen. As before, equation (9) can then be used to express the counterexample in terms of a, b, and c.

Example 1.

Suppose we want to find a counterexample to the proposed inequality

$$s^2 \le r^2 + 4rR + 4R^2.$$

We apply Algorithm B. The R-r-s form is

$$r^2 + 4rR + 4R^2 - s^2 \ge 0.$$

The x-y form is

$$x^2 - y^2 + 4x + 4 \ge 0.$$

The t form is

$$t^4 - 6t^2 + 1 \ge 0.$$

Polynomial inequalities with one variable are well-understood. The following plot shows that $t^4 - 6t^2 + 1$ can be negative when $t > 1$.

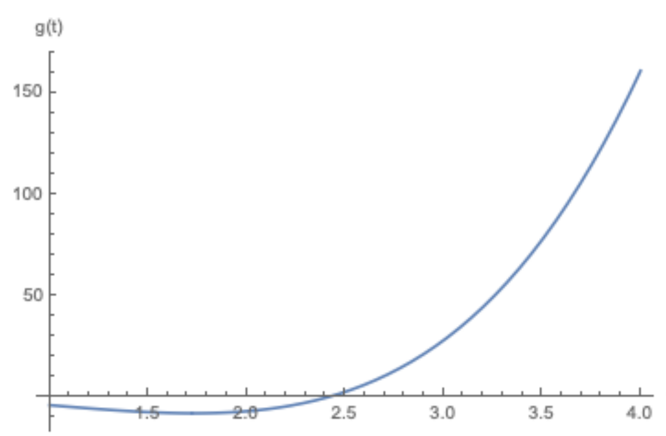


FIGURE 4. Plot of $g(t) = t^4 - 6t^2 + 1$

But to use the method described above, we use the Mathematica command

```
FindInstance[t^4-6t^2+1<0 && t>1, t]
```

which returns $t = 2$ as a counterexample. Equation set (7) then give $x = 12/25$ and $y = 64/25$. Equation set (8) then give $R = 25$, $r = 12$, and $s = 64$ as our counterexample.

If we want to present the counterexample in terms of a, b, and c, we issue the commands

```
equationForabc = x^3-2s*x^2+(r^2+s^2+4R*r)x-4R*r*s == 0;
Solve[equationForabc/.{R->25, r->12, s->64}, x]
```

which tells us that a, b, and c are 40, 40, and 48. Scaling down, we can say that the triangle with sides 5, 5, and 6 is a counterexample.

Now let's work through an example where the degree of the xy-polynomial is larger than 2, so that the t-form cannot be used.

Example 2.

Suppose we want to find a counterexample to the proposed inequality

$$\frac{26}{5R+12r} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

We apply Algorithm B. The R-r-s form is

$$\frac{12r^3 + 53r^2R + 20rR^2 - 104rRs + 12rs^2 + 5Rs^2}{48r^2Rs + 20rR^2s} \geq 0.$$

The x-y Form is

$$\frac{12x^3 + 53x^2 + 12xy^2 - 104xy + 20x + 5y^2}{4\left(12x^2 + 5x\right)y} \geq 0.$$

After clearing of fractions (step 5), we find that the inequality is equivalent to

$$12x^3 + 53x^2 + 12xy^2 - 104xy + 20x + 5y^2 \geq 0.$$

The polynomial on the left is of degree 3, so we cannot use the t-Form.

We use the Mathematica command

```
FindInstance[12 x^3+53 x^2+12 x y^2-104 x y+20 x+5 y^2<0
        && x>0 && y>0 && fundamentalFunction <= 0, {x,y}]
```

which returns the instance $x = 1/8$ and $y = 13/8$. Without loss of generality, let $R = 8$, so that $R = 8$, $r = 1$, and $s = 13$ is our counterexample. The equation for a, b, and c becomes

$$x^3 - 26x^2 + 202x - 416 = 0,$$

so the values for a, b, and c that give a counterexample are the roots of this equation. These are $a \approx 3.24686$, $b \approx 10.2352$, and $c \approx 12.5179$.

9. Finding Best Inequalities with One Parameter

Suppose we want to find a triangle inequality involving some undetermined constant k, and we want to find the best value for k. For example, suppose we want to find an inequality of the form

$$s \leq 2R + kr$$

where k is some constant. Suppose further that we want the "best" value for k that makes this inequality true (i.e., in this case, we want the smallest possible value for k).

If $\{f_i \geq 0\}$ is a set of inequalities, we say that $f \in \{f_i\}$ is the best inequality of the set if $f \geq f_i$ for all i. If $\{f(k) \geq 0\}$ is a set of inequalities involving a parameter k, we say that k_0 is the *best constant* k if $f(k_0) \geq f(k)$ for all k or if $f(k_0) \leq f(k)$ for all k. See [20, p. 43].

Algorithm K [Find Best Constant k]

Find the best value for k for a given inequality containing the parameter k.

INPUT: We are given an inequality $f(k) \geq 0$ involving a real parameter k and symmetric homogeneous rational functions of variables and functions listed in Section 4.

STEP 1: Apply Algorithm B, but stop at step 7, when we are about to check if $f(x, y) \geq 0$. In this case, the inequality is actually of the form $f(x, y, k) \geq 0$.

STEP 2: Use the following Mathematica commands to determine the set of values for k that make the inequality true, where `expr`$= f(x, y, k)$.

```
fundamentalFunction = (x^2+y^2)^2+12x^3-20x*y^2+48x^2-4y^2+64x;
fundamentalRegion = fundamentalFunction <= 0 && x >= 0 && y >= 0;
Resolve[ForAll[{x, y}, fundamentalRegion, expr >= 0]]
```

`Resolve` eliminates ∀ and ∃ quantifiers from an expression. According to [36], `Resolve[expr]` can in principle always eliminate quantifiers if `expr` contains only polynomial equations and inequalities over the reals.

OUTPUT: The conditions that k must satisfy to make the given inequality true.

Example 1.

Let's apply Algorithm K to the inequality $s \leq 2R + kr$.

In this case, Algorithm K returns $4 + k \geq 3\sqrt{3}$. Thus, the smallest value for k is $3\sqrt{3} - 4$. This agrees with the value given in [20, p. 47].

We could also check that this value is plausible by using Algorithm B. Algorithm B reports that the inequality $s \leq 2R + (3\sqrt{3} - 4)r$ is true but the inequality $s \leq 2R + (3\sqrt{3} - 4 - 1/1000)r$ is false.

Example 2.

For a more complicated example, we look at the inequality

$$\frac{11\sqrt{3}}{5R+12r+k(2r-R)} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

considered by Wu in [39]. In this case, the x-y form is

$$(2kx-k+12x+5)\times$$
$$(2kx^3+7kx^2+2kxy^2-4kx-ky^2+12x^3+53x^2+12xy^2-44\sqrt{3}xy+20x+5y^2)$$

and Algorithm K returns the fact that k must be less than or equal to the root of the equation

$$405x^5+6705x^4+129586x^3+1050976x^2+2795373x-62181=0$$

that is approximately 0.0220608. This agrees with the result found by Wu.

Example 3.

In this example, the variable k occurs more than once in the inequality.

Chirciu [11] found the inequality

$$\left(\sum r_a\right)\left(\sum \frac{1}{r_a}\right)+\frac{2kr}{R} \ge (k+9)\prod \frac{a+b}{2c} \qquad \text{for } k \le \frac{37}{11}.$$

To find the best value for k we apply Algorithm K which returns the fact that

$$k \le \frac{175+16\sqrt{94}}{81}.$$

This gives a stronger inequality.

Theorem 1. *The following inequality holds for all triangles.*

$$\left(\sum r_a\right)\left(\sum \frac{1}{r_a}\right)+\frac{2kr}{R} \ge (k+9)\prod \frac{a+b}{2c} \qquad \text{for } k \le \frac{175+16\sqrt{94}}{81}.$$

10. Finding Conditions for Equality

If an inequality is true, we can find the conditions for which equality holds using the following algorithm.

Algorithm Q [Find Conditions for Equality]

Algorithm Q determines all triangles for which a given inequality becomes an equality.

INPUT: An inequality involving symmetric homogeneous rational functions of variables and functions listed in Section 4.

STEP 1: Apply Algorithm B, but stop at step 7 when we are about to check if $f(x, y) \geq 0$.

STEP 2: Change the inequality symbol to an equality symbol.

STEP 3: [Check the polynomial equality $f(x, y) = 0$]

Use Mathematica to solve the equation using the commands

```
Solve[f[x,y]==0 && fundamentalRegion,{x,y},PositiveReals]
```

which finds all (x, y) in the fundamental region for which the equality is true.

STEP 4: [Find R, r, and s]

A triangle satisfying the equality is unique only up to a scale factor, so we can pick any value we want for R. For simplicity, pick $R = 1$.

For each (x, y) found in Step 3, find the corresponding values for R, r, and s using equation set (10).

$$R = 1$$
$$r = x$$
$$s = y$$

Each triple (R, r, s) so found represents a triangle for which equality holds in the given inequality.

STEP 5: (optional) [Find a, b, and c]

If you want to describe the triangle in terms of its sides rather than in terms of R, r, and s, proceed as follows. The values of a, b, and c are the roots of the equation

$$x^3 - 2sx^2 + (r^2 + s^2 + 4rR)x - 4rRs = 0$$

as we saw in Algorithm C.

OUTPUT: The set of triangles for which the given inequality becomes an equality.

Example 1: Let us use AlgorithmQ to find the conditions for equality in the inequality

$$\frac{11\sqrt{3}}{5R + 12r} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Step 1 gives

$$20x + 53x^2 + 12x^3 - 44xy\sqrt{3} + 5y^2 + 12xy^2 \geq 0.$$

Step 2 changes this to an equality.
Step 3 solves for x and y and gets $(x,y)=\left(\frac{1}{2},\frac{3\sqrt{3}}{2}\right)$ as the only real solution pair in the fundamental region.
Step 4 gives $R=1$, $r=\frac{1}{2}$, and $s=\frac{3\sqrt{3}}{2}$.
Step 5 solves the equation $x^3-3\sqrt{3}x^2+9x-3\sqrt{3}=0$ and finds that the three roots are all $\sqrt{3}$. Thus, equality only holds for an equilateral triangle.

We already saw in Figure 3 what the regions look like. The boundary of the inequality regions touched the boundary of the fundamental region at the two points, $(0,0)$. and $\left(\frac{1}{2},\frac{3\sqrt{3}}{2}\right)$ The point $(x,y)=(0,0)$ represents a degenerate triangle and $(1/2,3\sqrt{3}/2)$ represents an equilateral triangle.

Example 2:

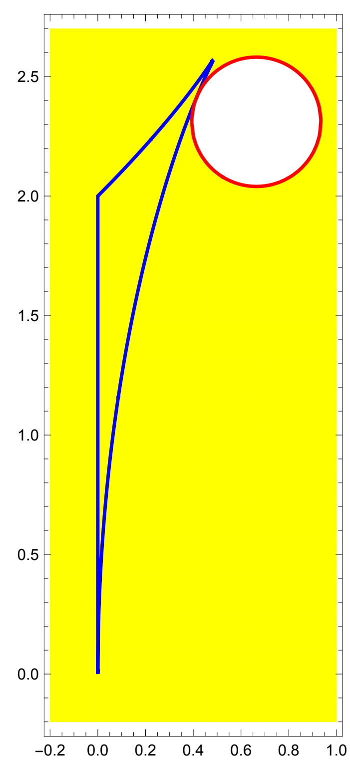
Many inequalities have the property that equality holds if and only if the triangle is equilateral. We now give an example of an inequality where this is not the case. Let us see how Algorithm Q works with the inequality

$$2r^2+\left(3-4\sqrt{2}\right)rR+\left(10+\sqrt{2}\right)R^2-\left(5+3\sqrt{2}\right)Rs+2s^2\geq 0.$$

The xy-Form is

$$2x^2+\left(3-4\sqrt{2}\right)x+2y^2-\left(5+3\sqrt{2}\right)y+\sqrt{2}+10\geq 0.$$

After changing this to an equality and solving for (x,y), we find that the only solution pair in the fundamental region is $(x,y)=(\sqrt{2}-1,\sqrt{2}+1)$. This gives rise to $R=1$, $r=\sqrt{2}-1$, and $s=\sqrt{2}+1$. Converting to abc-Form, we find $\{a,b,c\}=\{2,\sqrt{2},\sqrt{2}\}$. Scaling, we see that the only triangle for which equality holds is a triangle similar to one with sides of lengths 1, 1, and $\sqrt{2}$. That is, equality occurs if and only if the triangle is an isosceles right triangle.



In the figure to the right, the blue curve is the boundary of the fundamental region. The inequality region is yellow and is the regions outside the circle with the red circumference. The boundaries of the two regions are tangent at the point $(x,y)=(\sqrt{2}-1,\sqrt{2}+1)$, which is the point corresponding to isosceles right triangles.
The inequality is true because every point in the fundamental region lies in the yellow region. The only time equality holds is at the point where the two boundaries are tangent.

Note. The graphical view of an inequality shows that it is possible to have an inequality in which equality holds for two (or more) shapes of triangles. Just create two (or more) regions externally tangent to the hypocycloid bounding the fundamental region and form the inequality corresponding to the exterior of these regions.

Example 3:

Often, the shape of a triangle for which equality holds has sides whose lengths are very complicated. Mathematica has no problem with expressions involving roots of polynomials.

Let us consider the following Bonnesen-like inequality, which comes from [23]. Recall that $L = 2s$.

$$L^2 - 12\sqrt{3}K \geq kr(R - 2r)$$

First we use Algorithm K to find the largest value of k for which this inequality is true. Algorithm K tells us that the best value for k is the root of the equation

$$x^3 - 280x^2 + 10368x - 62208 = 0$$

that is closest to 35. Let us call this root k_0. Algorithm B confirms that the inequality

$$L^2 - 12\sqrt{2}K \geq k_0 r(R - 2r)$$

is true. We will now use Algorithm Q to determine when equality holds.

The xy-Form (changed to an equality) is

$$4y^2 - 12\sqrt{3}xy - k_0 x(1 - 2x) = 0.$$

The output from the `Solve` call in Step 3 finds two pairs (x, y) that satisfy the equality and lie in the fundamental region.

The first solution is

$$(x, y) = \left(\frac{1}{2}, \frac{3\sqrt{3}}{2}\right).$$

This corresponds to the set of equilateral triangles.

The second solution is

$$(x, y) = (x_2, y_2)$$

where x_2 and y_2 are given as root expressions. Setting $R = 1$, $r = x_2$, and $s = y_2$, we then find that a, b, and c are the roots of the equation

$$x^3 + c_2 x^2 + c_1 x + c_0 = 0$$

where c_2, c_1, and c_0 are also given as root expressions. Finally, Mathematica solves this cubic and gives

$$(a, b, c) = (1, 1, \lambda)$$

where λ is the largest real root of the equation $31x^3 - 28x^2 - 16x + 4 = 0$.

(((to be fixed))) The t-Form (with a constant factor removed) is

$$(k_0 - 64)t^6 + 96\sqrt{3}t^5 - 7k_0 t^4 - 96\sqrt{3}t^3 + 15k_0 t^2 - 9k_0 = 0.$$

Solving for t, Mathematica finds two real solutions for t larger than 1. One of them, after simplifying (using `RootReduce`) is $t = \sqrt{3}$, which we have seen corresponds to the set of equilateral triangles. The other solution is the largest real root of the equation

$$27t^6 - 163t^4 + 225t^2 - 81 = 0.$$

Call this value t_0. Converting to abc-Form shows that

$$\{a, b, c\} = \{8t_0(t_0^2 - 1), 4t_0(t_0^2 + 1), 4t_0(t_0^2 + 1)\}.$$

Scaling this shows that the triangle is similar to the triangle with sides 1, 1, and $2(t_0^2 - 1)/(t_0^2 + 1)$. Simplifying again (using `RootReduce`) lets us say that the inequality holds for any triangle similar to the isosceles triangle with sides 1, 1, and λ, where λ is the largest real root of the equation $31x^3 - 28x^2 - 16x + 4 = 0$.

Theorem 2. *There exists an inequality that is true for all triangles and a point where equality holds lies strictly inside the fundamental region.*

(((proof to be inserted here)))

Theorem 3. *There are homogeneous non-strict inequalities involving polynomial functions of R, r, and s that are true for all isosceles triangles, but are not true for all triangles.*

Proof. Consider the inequality

$$400(r^2 + s^2) + 1027R^2 \geq 80R(r + 16s).$$

Algorithm B shows that this inequality is false. A plot of a portion of the inequality region is shown in Figure 5. The boundary of the inequality region is colored red. The inequality region consists of all points in the plane outside the red oval. This region is colored yellow. The fundamental region is outlined in blue. Note that the inequality region has a hole in it that lies inside the fundamental region. This hole is colored white. Points inside this hole correspond to triangles that do not satisfy the inequality. But note that every point on the boundary of the fundamental region satisfies the inequality. Thus, all isosceles triangles satisfy the inequality, but not all triangles satisfy the inequality. $\square$

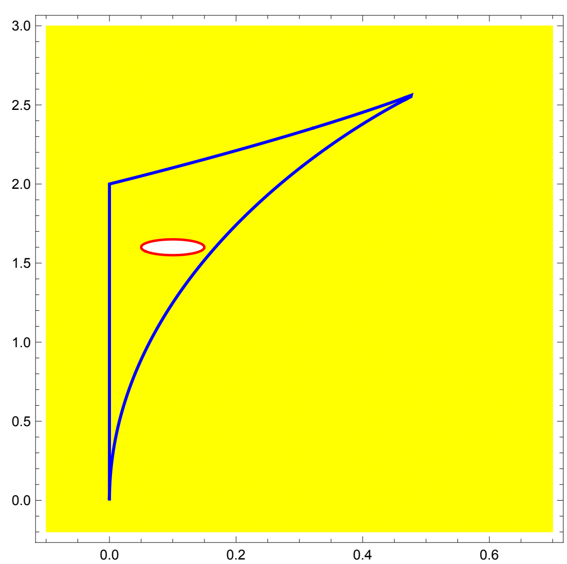


FIGURE 5. Plot of Inequality Region and Fundamental Region

Acknowledgment. I would like to thank Ercole Suppa for finding a simpler counterexample for use with Theorem 3.

11. Mistakes in the Literature

The literature on triangle inequalities and identities is so vast that it is inevitable that some mistakes could get published. While testing Algorithm Blundon, we ran across a number of errors in the literature. We list below the errors found.

• The inequality $27\prod(b^2+c^2-a^2)^2 \le (4K)^6$ appears in Bottema [4, p. 47]. This is incorrect as noted in [33]. The correct result is the following.

Theorem 4 (Ono's Inequality)**.** *For any acute triangle,*

$$27\prod(b^2+c^2-a^2)^2 \le (4K)^6.$$

• The inequality $4R(R-4K/s)^3 \ge (s^2/4+4K^2/s^2-2R^2-5KR/s)^2$ appears in Bottema [4, p. 71]. This is incorrect. A counterexample is $r=1$, $s=24$, $R=16$. The correct inequality is given in [20, p. 3–4].

Theorem 5 (Nakajima's Inequality)**.** *For any triangle,*

$$4R\left(R-\frac{2K}{s}\right)^3 \ge \left(s^2+\frac{K^2}{s^2}-2R^2-\frac{10KR}{s}\right)^2.$$

• The inequality $\sum \frac{1}{a(b+c-a)} \ge \frac{r}{8R}\left(5-\frac{9r}{4R+r}\right)$ appears in Crux Mathematicorum [19, p. 101]. This inequality is dimensionally incorrect and hence false. The correct inequality is the following.

Theorem 6. *For any triangle,*

$$\sum \frac{1}{a(b+c-a)} \ge \frac{1}{8rR}\left(5-\frac{9r}{4R+r}\right).$$

• The inequality $(a^2+b^2+c^2)/(ab+bc+ca) \ge 2R/(R+2r)$ appears in the Romanian Mathematical Magazine [22]. This is incorrect. The correct inequality is the following.

Theorem 7. *For any triangle,*

$$\frac{6}{5}\frac{R}{R+2r} \le \frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{2R}{R+2r}.$$

• The inequality $a^2+b^2+c^2 \le 8R^2+8K^2/(27R^2)$ for all acute triangles appears in Mitrinovic [20, p. 243]. This is incorrect. The following are correct inequalities.

Theorem 8. *For any acute triangle,*

$$a^2+b^2+c^2 \le 8R^2+\frac{16K^2}{27R^2}.$$

Theorem 9. *For any acute triangle,*

$$a^2+b^2+c^2 \le \frac{17}{2}R^2+\frac{8K^2}{27R^2}.$$

• The inequality $\sum m_a^2 \le 6R^2+\sqrt{3}K/6$ for all acute triangles appears in Mitrinovic [20, p. 247]. This is incorrect. The correct inequality is the following.

Theorem 10. *For any acute triangle,*

$$\sum m_a^2 \le 6R^2 + \sqrt{3}K/3.$$

• The inequality $\prod a^2 \ge 4Rr(2R^2 + 8Rr + 3r^2)$ for all acute triangles appears in Mitrinovic [20, p. 248]. This is incorrect. The correct inequality is the following.

Theorem 11. *For any acute triangle,*

$$\prod a^2 \ge 4R^2r^2(2R^2 + 8Rr + 3r^2).$$

Ercole Suppa [28] has pointed out a number of errors in Alasia [1]. We used Algorithm R to correct these identities.

• Alasia [1, Formula 106] claims that $\sum a/(rr_a) = 2(2R+r_a)/K$. This is incorrect. A correct identity is the following.

Theorem 12. *For any triangle,*

$$\sum \frac{a}{rr_a} = \frac{2(4R+r)}{K}.$$

• Alasia [1, Formula 131] claims that $\sum r_a = 3R + r\sum \cos A$. This is incorrect. The correct identity is the following.

Theorem 13. *For any triangle,*

$$\sum r_a = 3R + R\sum \cos A.$$

• Alasia [1, Formula 406] claims that $\sum \sin A/\prod \sin A = R/r$. This is incorrect. The correct identity is the following.

Theorem 14. *For any triangle,*

$$\frac{\sum \sin A}{\prod \sin A} = \frac{2R}{r}.$$

• Alasia [1, Formula 421] claims that $2\prod \sin A = \sum \sin^2 A$. This is incorrect. A correct identity is the following.

Theorem 15. *For any triangle,*

$$4\prod \sin A = \sum \sin 2A.$$

• Alasia [1, Formula 427] claims that $\sum \cos 2A = 4\sin A \sin B \cos C - 1$. This is incorrect. A correct identity is the following.

Theorem 16. *For any triangle,*

$$\sum \sin 2A = 4\prod \sin A.$$

• Alasia [1, Formula 434] claims that $\sum(\tan A/\tan B + \tan B/\tan A) + 2 = \sum \sin A$. This is incorrect. A correct identity is the following.

Theorem 17. *For any triangle,*

$$\sum \left(\frac{\tan A}{\tan B} + \frac{\tan B}{\tan A}\right) + 2 = -\frac{4}{1 + \sum \cos 2A}.$$

12. Proving Identities

Since the quantities R, r, and s are independent, two symmetric homogeneous expressions inolving elements of a triangle will be identical if and only if their R-r-s forms are identical. We can formalize this as Algorithm V.

Algorithm V [Verifying Identities]

INPUT: Input to this algorithm is a conjectured identity involving symmetric homogeneous rational functions of elements of a triangle.

Step 1 [Convert to R-r-s Form]

Apply Algorithm R to both sides of the conjectured identity.

Step 2 [Compare Sides]

If the two results are identical, return `True`, else return `False`.

OUTPUT: Algorithm V returns true or false indicating the correctness of the conjectured identity.

Example

Theorem 18 (Suppa's Exradii Identity [27])**.** *In any triangle, the following identity holds.*

$$\sum ab = \sum r_a(r_b + r) \tag{11}$$

Proof. Applying Algorithm R, the left side of (11) is equivalent to $r^2 + 4rR + s^2$. The right side of (11) is also equivalent to $r^2 + 4rR + s^2$. Thus, (11) is an identity. □

13. Proving Inequalities for Special Triangles

The following theorem is well known [4, p. 102].

Theorem 19.
(a) A triangle is acute if and only if $s > 2R + r$.
(b) A triangle is obtuse if and only if $s < 2R + r$.
(c) A triangle is a right triangle if and only if $s = 2R + r$.

This allows us to find inequalities restricted to these classes of triangles.

For example, in Algorithm B, instead of using the code

```
fundamentalFunction = (x^2+y^2)^2+12x^3-20x*y^2+48x^2-4y^2+64x;
fundamentalRegion = fundamentalFunction<=0 && x>0 && y>0;
truth = Simplify[f[x,y] >= 0, fundamentalRegion];
```

we can use the following code instead to prove inequalities that are true just for acute triangles.

```
fundamentalFunction = (x^2+y^2)^2+12x^3-20x*y^2+48x^2-4y^2+64x;
fundamentalRegion = fundamentalFunction<=0 && x>0 && y>0;
acuteRegion = y>x+2;
truth = Simplify[f[x,y] >= 0, fundamentalRegion && acuteRegion];
```

Note that the R-r-s condition $s > 2R + r$ becomes the x-y condition $y > x + 2$ by applying the substitutions $r \to xR$ and $s \to yR$, since $R \neq 0$.

If we plot the line $y = x + 2$ (in red) on the same set of axes as the fundamental region, we get the following graph:

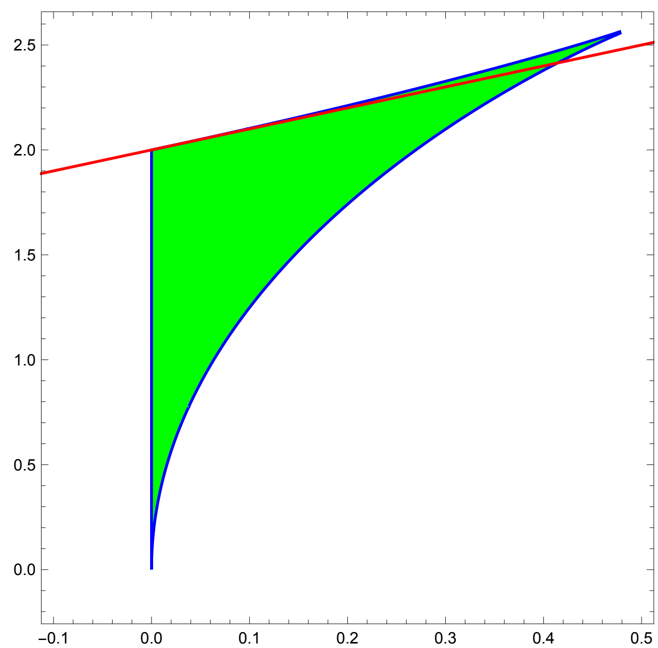


FIGURE 6. Split of Fundamental Region into Acute and Obtuse Regions

The portion of the fundamental region (green) above the red line corresponds to acute triangles. The portion below the red line corresponds to obtuse triangles.

An inequality will be true for all acute triangles if the inequality region covers the portion of the green region above the red line (the *acute region*).

If using the t-Form, the t condition corresponding to $y > x+2$ is $4t^3+1 > t^4+4t^2$. Since $t > 1$, this is equivalent to $t^2 < 2t + 1$ or $1 < t < 1 + \sqrt{2}$.

Example.

Let us work through proving that the inequality

$$\sum \tan A \geq 3\sqrt{3}$$

is true for all acute triangles using this method.

Applying Algorithm B, we find that the R-r-s Form is

$$\frac{-3\sqrt{3}r^2 - 2r\left(6\sqrt{3}R + s\right) + 3\sqrt{3}\left(s^2 - 4R^2\right)}{r^2 + 4rR + 4R^2 - s^2} \geq 0.$$

The x-y polynomial (from Step 5) is

$$\left(x^2 + 4x - y^2 + 4\right)\left(-3\sqrt{3}x^2 - 2xy - 12\sqrt{3}x + 3\sqrt{3}y^2 - 12\sqrt{3}\right).$$

For step 7, we issue the following code:

```
fundamentalFunction = (x^2+y^2)^2+12x^3-20x*y^2+48x^2-4y^2+64x;
fundamentalRegion = fundamentalFunction<=0 && x>0 && y>0;
acuteRegion = y>x+2;
inequalityRegion = (4+4x+x^2-y^2)(-12Sqrt[3]-12*Sqrt[3]x
                   -3Sqrt[3]x^2-2x*y+3Sqrt[3]y^2) >= 0;
Simplify[inequalityRegion, fundamentalRegion && acuteRegion]
```

The result of the `Simplify` command is `True`, which proves the inequality.

Incidentally, if we apply Algorithm Q, we find that amongst all acute triangles, equality occurs if and only if $(x, y) = \left(\frac{1}{2}. \frac{3\sqrt{3}}{2}\right)$. That is, equality occurs if and only if the triangle is equilateral.

Had we not known what the best constant was for the right side of the inequality, we could have applied Algorithm K to the inequality

$$\sum \tan A \geq k.$$

Step 2 of Algorithm K would be modified by changing `fundamentalRegion` to `fundamentalRegion && acuteRegion` in order to restrict it to acute triangles. The output from Algorithm K is then $3\sqrt{3}$, telling us that this is the best possible constant.

Other shape triangles.

We can prove inequalities for other shape triangles by replacing the fundamental region by the region formed by intersecting the fundamental region and the region corresponding to the shape of triangle we are interested in.

Given a condition on a, b, and c, the R-r-s condition is found by applying Algorithm R. The xy-condition is then found using equation set (10) and canceling R factors.

14. Proving Trivariate Inequalities

Algorithm B can be used to prove symmetric homogeneous trivariate inequalities involving rational functions of x, y, and z subject to the constraints $x > 0$, $y > 0$, and $z > 0$ by using the following algorithm.

We use the following theorem [5, Section 2] which expresses the correspondence between positive inequalities and triangle inequalities.

Theorem 20 (The Fundamental Correspondence)**.** *Let a, b, c be the sides of a triangle. Then the inequality $f(a, b, c) \geq 0$ is equivalent to the inequality $g(x, y, z) = f(\frac{y+z}{2}, \frac{z+x}{2}, \frac{x+y}{2}) \geq 0$ for all $x, y, z > 0$ where*

$$\begin{aligned} x &= b + c - a, \\ y &= c + a - b, \\ z &= a + b - c. \end{aligned}$$

Variables a, b, and c can be expressed in terms of x, y, and z via the equations

$$\begin{aligned} a &= \frac{y+z}{2}, \\ b &= \frac{z+x}{2}, \\ c &= \frac{x+y}{2}. \end{aligned}$$

Algorithm XYZ [Prove Trivariate Inequality]

INPUT: Input to this algorithm is an inequality involving symmetric homogeneous rational functions of x, y, and z subject to the constraints $x > 0$, $y > 0$, and $z > 0$.

Step 1 [Convert to a, b, c]

Make the following substitutions.

$$\begin{aligned} x &\to b + c - a, \\ y &\to c + a - b, \\ z &\to a + b - c. \end{aligned}$$

Step 2 [Apply Algorithm B]

By Theorem 20, the resulting inequality is equivalent to the given one with the constraints $a > 0$, $b > 0$, $c > 0$, $a + b > c$, $b + c > a$, $c + a > b$.

So apply Algorithm B to determine whether the inequality is true or false.

OUTPUT: Algorithm XYZ returns true or false indicating the correctness of the conjectured identity.

Application.

We can combine Algorithm K with Algorithm XYZ to find the best k that makes the following inequality true for all $x, y, z > 0$.

$$\frac{x^5+y^5+z^5}{(xyz)^2} - \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq k\left(\frac{x^2+y^2+z^2}{xyz} - \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\right) \tag{12}$$

We get the following result.

Theorem 21. *Inequality (12) is true for all $x, y, z > 0$, whenever*

$$k \leq \frac{1}{3}\left(8+\sqrt[3]{\frac{1}{2}\left(997-69\sqrt{69}\right)}+\sqrt[3]{\frac{1}{2}\left(997+69\sqrt{69}\right)}\right).$$

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