International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 3, 2018, pp.1-4.
Received 10 January 2018. Published on-line 24 January 2018
web: http://www.journal-1.eu/
©The Author(s) This article is published with open access[1].

A Proof of Dao's Generalization of the Sawayama Lemma

Nguyen Chuong Chi
Rowna 4 str., 05-075 Warsaw, Poland
e-mail: Nguyenchuongchi@yahoo.com

Abstract. We give a proof of Dao's generalization of the Sawayama lemma.

Keywords. Euclidean Geometry, Sawayama-Thébault's Theorem, Dao's circle

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The Sawayama-Thebault Theorem is an exceptionally elegant theorem in Euclidean geometry. The theorem is as follows:

Theorem 1.1. *Given a triangle ABC, construct its circumcircle (O). Now take any point D on side BC and draw line AD. Now construct the 2 circles that are internally tangent to (O), BC, and AD, with centers O_1 and O_2 respectively. Then O_1, O_2 and the incenter of $\triangle ABC$ are collinear. (Figure 1)*

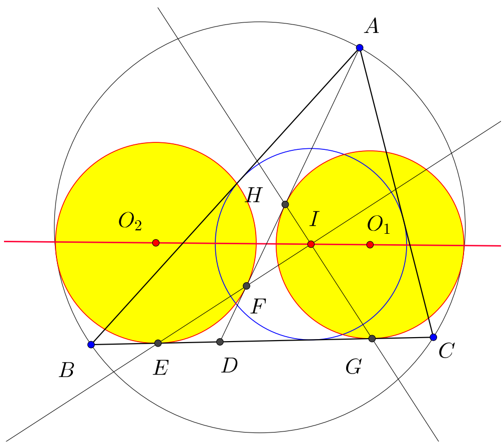

Figure 1.

[1]This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

The Sawayama-Thebault Theorem was a problem no.3887 in American Mathematical Monthly. This problem was published in 1938 and only in 1973 the first solution of 24 pages in print, was proposed by K. B. Taylor. Since then, several other solutions have appeared; these solutions were always numerical, eg. R. Shail, Stark or Gerhard Turnward, to name just a few authors [2].

However, Jean-Luis Ayme, a French professor of Mathematics, discovered in 2003 that this theorem had been published earlier, also in American Mathematical Monthly, as early as 1905, by Y. Sawayama, a Japanese teacher from Tokyo [2][3]. That is why now the theorem is named after the two mathematicians. The main tool for the proof of the Sawayama-Thebault Theorem is Sawayama's lemma.

Lemma 1.1. *(Sawayama-*[3]*) Given a triangle ABC, take any point D on the side BC. Now construct a circle that is tangential internally to the circle circumscribed on the triangle ABC, and tangential to BC and AD in points E and F respectively. Then E, F and the incenter of ABC are colinear. (Figure 1)*

When talking about the Sawayama lemma, a special case of the theorem should be mentioned, namely, the Nixon theorem. The case when line AD is coincidental with AB or AC. The circle in Sawayama's lemma is now called a mixtilinear circle. The Nixon theorem, which explains one property of this circle, goes as follows:

Theorem 1.2. *(Nixon* [4]*). The mixtilinear circle corresponding to point A, tangential to AB, AC in points M, N respectively. Then the incenter of ABC is the midpoint of segment MN.*

Not long ago Dao Thanh Oai, a Vietnamese engineer, proposed a generalization of the Sawayama lemma, which at the same time is a generalization of the Sawayama-Thebault theorem. The Dao's generalization of the Sawayama lemma as follows:

Theorem 1.3. *(Dao-*[6]*). Given a triangle ABC and P, Q which are isogonal conjugate. AP and AQ cut the circumcircle of triangle ABC in points D and E. Any two lines going through D and E cut (ABC) in two points T and N, and the line BC in two points G and H. Lines PG and HQ cut the circle $(GHNT)$ in points K and F. Then, K, F and A are collinear.*

When in the research Theorem 1.3 Dao Thanh Oai defined the circle, an American professor of Mathematics C. Kimberling described Dao's circle [7] as follows:

Theorem 1.4. *(Dao's circle-*[8]*). Let P is a point in the plane of a triangle ABC, but not on a sideline (BC, CA, AB). Let P' be the isogonal conjugate of P. Let (O) be the circumcircle of ABC, and let $C(P)$ be the conic through A, B, C, P, P'. Let D be the point in $(O) \cap C(P)$ other than A, B, C; let E be a point on (O), other than A, B, C, D, and let E' the point in $DE \cap C(P)$, other than D. The points P, P', E, E' lie on a circle, here named the Dao circle of P.*

You can see Dao's circle in Encyclopedia of Triangle Centers, centers $X(10097) - X(10103)$. You can see another generalization of Sawayama's lemma in [9]. In the following part, I am presenting my own geometrical proof of Theorem 1.3.

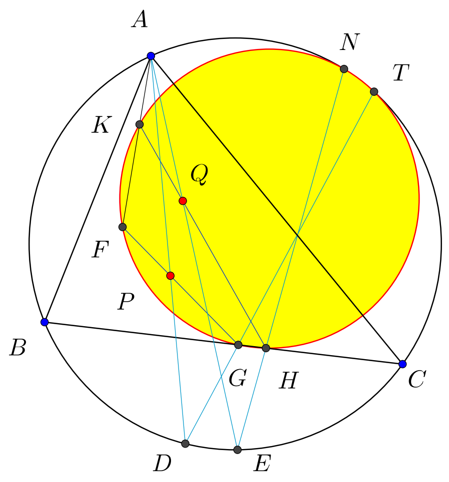

FIGURE 2. Dao's generalization of the Sawayama's lemma

2. PROOF OF THE DAO'S GENERALIZATION SAWAYAMA LEMMA

We will prove that points A, K, F are collinear through proving, that $\angle PFA = \angle PFK$ (Figure 3).

First and foremost, in the body of the problem a $GHNT$ circle was mentioned, while there is no indication of a quadrangle $GHNT$ inscribed in the circle. In fact, this thesis is always true, because points P, Q are isogonal conjugate, which causes $DE \parallel BC$, hence $\angle DEN = \angle GHN$, so $\angle GTN + \angle GHN = \angle GTN + \angle DEN = 180^0$, which means, that $GHNT$ is inscribed in a circle.

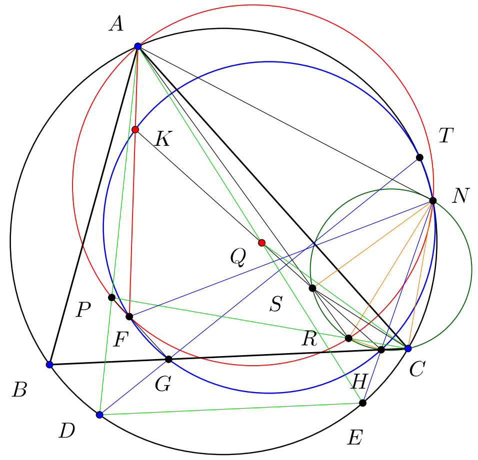

FIGURE 3.

Among the inscribed angles, we have $\angle PFN = \angle GHN = \angle DEN$, therefore $\angle NAP + \angle PFN = \angle NAD + \angle DEN = 180^0$, hence the quadrangle APFN is inscribed in the circle.

Let us take R which is the point of intersection of the line CP with the circle $(APFN)$. Again, taking into account inscribed angles, we have $\angle PRN =$

$\angle PFN = \angle GHN$, therefore$\angle CRN = \angle CHN$, so the quadrangle $CHRN$ is inscribed in the circle. Now let us take S which is the point of intersection of the line HK (other than H) with the circle $(CHRN)$.

Now, we have $\angle RSH = \angle RCH = \angle PCB = \angle QCA$ (1) (because P, Q are isogonal conjugate).

On the other hand, $\angle HSC = \angle HNC = \angle ENC = \angle EAC = \angle QAC$, hence the quadrangle $QACS$ is inscribed in the circle, therefore we can deduce that $\angle QSA = \angle QCA$ (2).

Comparing (1) with (2), we have $\angle RSH = \angle QSA (= \angle QCA)$, and at the same time Q, S, H are collinear, hence A, S, R must also be collinear.

Then, again, the inscribed angles: $\angle PFA = \angle PRA = \angle CNS = \angle GHS = \angle GHK = \angle PFK$ - Hence, finally: $\angle PFA = \angle PFK$.

The last equality is our desired solution, so we can directly deduce that A, K, F are collinear. The new generalization Sawayama's lemma proposed by Dao Thanh Oai is now fully proved.

REFERENCES

[1] V. Thébault, Problem 3887, Three circles with collinear centers, Amer. Math. Monthly,45 (1938) 482–483.

[2] Jean-Louis Ayme, Sawayama and Thebault's theorem, 225–229.

[3] Y. Sawayama, A new geometrical proposition, Amer. Math. Monthly, 12 (1905) 222–224.

[4] R. C. J. Nixon, Question 10693, Reprints of Educational Times, London (1863-1918) 55 (1891) 107.

[5] O.T.Dao, Advanced Plane Geometry, message 3349, July 21, 2016.

[6] O.T.Dao, A Generalization of Sawayama and Thébault's Theorem, International Journal of Computer Discovered Mathematics pp.33-35, available at `http://www.journal-1.eu/2016-3/Dao-Thanh-Oai-Sawayama-Thebault-pp.33-35.pdf`

[7] C. Kimberling, P. Moses, centers X(10097)-X(10103), available at `http://faculty.evansville.edu/ck6/encyclopedia/ETCPart6.html#X10097`

[8] T. O. Dao, Advanced Plane Geometry, message 3383, July 31, 2016. `https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/3383`

[9] Dao Thanh Oai, Another Generalization of the Sawayama and Thébault's Theorem, pp.36-39. International Journal of Computer Discovered Mathematics pp.33-35. available at `http://www.journal-1.eu/2016-3/Dao-Thanh-Oai-Another-Generalization-pp.36-39.pdf`