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An Upper Estimate for the Fermat-Torricelli Ratio

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Abstract. A *Steiner star* for a set $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ of n points in $\mathbb{R}^2$ connects an arbitrary point in $\mathbb{R}^2$ to all points of $\mathcal{P}$, while a *rooted star* connects one of the points in $\mathcal{P}$ to the remaining $n-1$ points of $\mathcal{P}$. All connections are realized by straight line segments. Let the length of a star be the total Euclidean length of its edges. The Steiner star of minimum length is known as the *Fermat-Torricelli star*. Fekete and Meijer showed that the minimum rooted star is at most $\sqrt{2}$ times longer than the Fermat-Torricelli star for any finite point configuration in $\mathbb{R}^2$. The supremum of the ratio between these two lengths, over all finite point configurations in $\mathbb{R}^2$, is called the *Fermat-Torricelli ratio* of the plane, and it is conjectured to equal $4/\pi = 1.2732\ldots$. Fekete and Meijer's result was improved by Dumitrescu, Tóth and Xu, who lowered the bound to 1.3631, and later by Ismailescu and Park, who proved an upper estimate of 1.3546.

In this paper we report further progress and prove an upper bound of 1.3385. Our method relies on a result of Morgenstern which allows for a better estimate of the sum of distances determined by pairs of points from the set $\mathcal{P}$.

Keywords. Rooted Stars, Fermat-Torricelli Star, Facility Location, Geometric Optimization, Computer-Discovered mathematics.

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1. Introduction

Given $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ a set of n points in $\mathbb{R}^2$, for every $1 \leq i \leq n$, define the *star rooted at* P_i as the union of all straight-line segments joining the point P_i to all the other points in the set $\mathcal{P}$. A *Steiner star* is the union of all straight-line segments connecting some point in $\mathbb{R}^2$ to each point of $\mathcal{P}$.

The *Fermat - Torricelli point* of $\mathcal{P}$, also known as the *Weber point* or as the *geometric median* of $\mathcal{P}$, is the point in $\mathbb{R}^2$ that minimizes the sum of distances to the n given points of $\mathcal{P}$. See [15]. It is easy to show that if $\mathcal{P}$ is not the subset of a line, then the Fermat - Torricelli point is unique. Connecting the Fermat - Torricelli point of $\mathcal{P}$ to all the points of $\mathcal{P}$, we obtain the Steiner star of minimal length. We will call this the *Fermat -Torricelli star* of $\mathcal{P}$.

Obviously, the length of the Fermat-Torricelli star is no greater than the length of any of the rooted stars. As clarified in the next section we would like to quantify how much better the Fermat-Torricelli star can be when compared to any of the rooted stars. As we will see later on, one of the challenges we face is that locating the Fermat-Torricelli point exactly is not possible in general.

In 1643, Fermat [8] asked for the point that minimizes the sum of the distances to *three* given points. The first solution was obtained by Torricelli [13] using the focal property of the ellipse.

If the triangle determined by the three given points has an angle greater than 120°, the Fermat -Torricelli point is located at the obtuse-angled vertex. Otherwise, the Fermat-Torricelli point is the interior point of the triangle such that the angles determined by the rays through the three points are all equal to 120°.

If $n = 4$, the situation is much simpler: if the points of $\mathcal{P}$ form a convex quadrilateral, then the Fermat-Torricelli point is the point of intersection of the diagonals. Otherwise, the Fermat-Torricelli point is the point of $\mathcal{P}$ which lies in the interior of the convex hull of $\mathcal{P}$.

However, for all $n \geq 5$, it has been shown that no explicit formula, nor an exact algorithm involving only arithmetic operations and radicals, can exist in general. Cockayne and Melzak [3] proved that the Fermat -Torricelli point for a five-point set cannot be constructed with ruler and compass. Using tools from Galois theory, Bajaj [1] showed that the Fermat-Torricelli problem is in general not solvable by radicals. The Fermat-Torricelli point can however be located with arbitrary precision, using for instance Weiszfeld's algorithm. See e. g. [2, 16].

Although in general an explicit formula for the location of the Fermat-Torricelli point set does not exist, it is possible to describe this point implicitly. The following result is mentioned in most references dealing with the Fermat-Torricelli problem.

Theorem 1.1. [4, 7, 10] *Let O be the Fermat-Torricelli point of a set $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$. Then*

$$\sum_{i=1}^{n} \frac{\overrightarrow{OP_i}}{OP_i} = \overrightarrow{0}. \tag{1}$$

2. Statement of the problem, definitions, notations and setup

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in $\mathbb{R}^2$ such that the Fermat-Torricelli center of $\mathcal{P}$ does not belong to $\mathcal{P}$. Assume that the center is located at the origin O. After an eventual scaling, we may assume that $\min\limits_{1\le i\le n} OP_i = 1$ and relabel the points such that $OP_i \le OP_j$, for all $1 \le i < j \le n$. This convention is going to be followed throughout the remainder of the paper.

Denote the intersection between the segment OP_i and the unit circle centered at O by Q_i, and denote $Q_iP_i = r_i$. Obviously, $0 = r_1 \le r_2 \le \ldots r_n$. Note that in this setting relation (1) can be written as $\overrightarrow{OQ_1} + \overrightarrow{OQ_2} + \ldots + \overrightarrow{OQ_n} = \overrightarrow{0}$ - see 1.

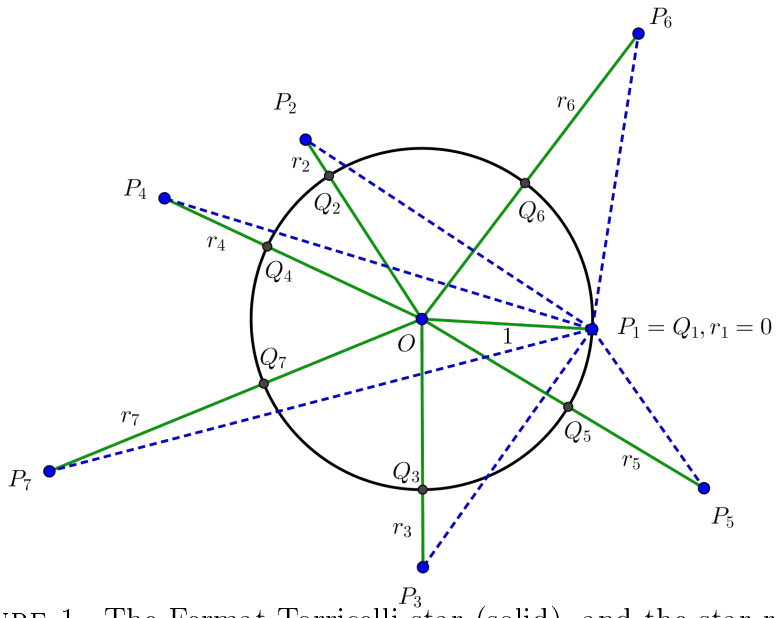

FIGURE 1. The Fermat-Torricelli star (solid), and the star rooted at P_1 (dashed).

For every $1 \le i \le n$, denote by $S_i(\mathcal{P})$ the length of the star rooted at P_i; that is, $S_i(\mathcal{P}) = \sum\limits_{j=1}^{n} P_iP_j$. Denote $S_{min}(\mathcal{P}) = \min\limits_{1\le i\le n} S_i(\mathcal{P})$.

The length of the Fermat-Torricelli star will be denoted by

$$FT(\mathcal{P}) = \sum_{i=1}^{n} OP_i = \sum_{i=1}^{n}(1 + r_i) = n(1 + R), \quad \text{where } R := \frac{1}{n}\sum_{i=1}^{n} r_i. \tag{2}$$

Finally, define

$$\alpha(\mathcal{P}) = \frac{S_{min}(\mathcal{P})}{FT(\mathcal{P})} \quad \text{and} \quad \alpha(n) = \sup_{|\mathcal{P}|=n} \alpha(\mathcal{P}). \tag{3}$$

The main question to be studied is the following:

Problem 2.1. *Find a good upper estimate for* $\alpha(n)$.

Fekete and Meijer [7] were the first to introduce this quantity, which they called the *Steiner ratio*. They proved that $\alpha(n) < \sqrt{2} = 1.4142\ldots$.

Dumitrescu, Tóth and Xu [5] improved this to $\alpha(n) \le 1.3630\ldots$. By refining their approach, Ismailescu and Park [9] further lowered this bound to $\alpha(n) \le 1.3545\ldots$.

It is conjectured that $\alpha(n) = 2\cot(\pi/2n)/n$, with equality if and only if the points $P_1, P_2, \ldots, P_n$ are the vertices of a regular n-gon. If true, this would imply that $\lim_{n\to\infty} \alpha(n) = 4/\pi = 1.2732\ldots$.

Note that there still is a relatively large gap between the conjectured value of $\alpha(n)$ and the current best upper bound. In this paper we further improve the upper estimate.

Theorem 2.1. *For every $n \geq 1$, we have $\alpha(n) \leq 1.3385$.*

This paper is organized as follows. In section 3 we present proofs of Fekete and Meijer, and Dumitrescu, Tóth and Xu bounds. These are going to be needed in section 4, where we will prove Theorem 2.1. In section 5 we consider the same problem with squared Euclidean distance instead of Euclidean distance.

3. Previous work

We start by listing several results that are going to be needed later. The first one shows that the sup in definition (3) of $\alpha(n)$ can be replaced by max.

Lemma 3.1. [7] *For every fixed n, there exists a point set $\mathcal{P}$ of n points in $\mathbb{R}^2$ for which $\alpha(\mathcal{P}) = \alpha(n)$. We will call such a set* optimal.

Proof. It suffices to consider those sets $\mathcal{P} \subset \mathbb{R}^2$ of cardinality n for which $FT(\mathcal{P}) = 1$ and the Fermat-Torricelli center is located at the origin O. But the set of these configurations is a compact subset of $\mathbb{R}^{2n}$. Since every $S_i(\mathcal{P})$ is a continuous function for every $1 \leq i \leq n$, then so is $S_{min}(\mathcal{P})$, which implies the result. □

Since we want to find an upper bound for $\alpha(n)$, it is enough to restrict our study to optimal sets. The next result states that all rooted stars of an optimal set have the same length.

Lemma 3.2. [7] *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be an optimal set as defined above. Then for every $1 \leq i < j \leq n$ we have that $S_i(\mathcal{P}) = S_j(\mathcal{P})$.*

Proof. Suppose that there exists $1 \leq k \leq n$ such that $S_k(\mathcal{P}) > S_{min}(\mathcal{P})$. Then for some $\epsilon > 0$ to be chosen later, replace the point P_k with a new point P_k' where $P_kP_k' = \epsilon$ and P_k' lies between O and P_k. Denote this new set by $\mathcal{P}'$. Obviously, the position of the Fermat-Torricelli point does not change. We have

$$FT(\mathcal{P}') - FT(\mathcal{P}) = OP_k' - OPk = -\epsilon, \quad \text{that is,} \quad FT(\mathcal{P}') = FT(\mathcal{P}) - \epsilon.$$

On the other hand, by the continuity of $S_k(\mathcal{P})$, for small enough ϵ, we are going to have that $S_k(\mathcal{P}') > S_{min}(\mathcal{P})$. Finally, for every $i \neq k$, we have that

$$S_i(\mathcal{P}) - S_i(\mathcal{P}') = P_iP_k - P_iP_k' \leq P_kP_k' = \epsilon, \quad \text{from which} \quad S_i(\mathcal{P}') \geq S_i(\mathcal{P}) - \epsilon.$$

Combining the last two inequalities, we obtain that $S_{min}(\mathcal{P}') \geq S_{min}(\mathcal{P}) - \epsilon$. Finally, we obtain

$$\frac{S_{min}(\mathcal{P}')}{FT(\mathcal{P}')} \geq \frac{S_{min}(\mathcal{P}) - \epsilon}{FT(\mathcal{P}) - \epsilon} > \frac{S_{min}(\mathcal{P})}{FT(\mathcal{P})},$$

which contradicts the optimality of $\mathcal{P}$. The proof is complete. □

The *average-length rooted star* is defined as

$$S_{avg}(\mathcal{P}) = \frac{1}{n}\sum_{i=1}^{n} S_i(\mathcal{P}) = \frac{2}{n}\sum_{1\le i<j\le n} P_iP_j. \tag{4}$$

We now introduce two quantities related to $\alpha(\mathcal{P})$. Let

$$\beta(\mathcal{P}) = \frac{S_1(\mathcal{P})}{FT(\mathcal{P})}, \quad \text{and} \quad \gamma(\mathcal{P}) = \frac{S_{avg}(\mathcal{P})}{FT(\mathcal{P})} = \frac{2\sum\limits_{1\le i<j\le n} P_iP_j}{n\,FT(\mathcal{P})}. \tag{5}$$

Recall that $S_1(\mathcal{P}) = \sum\limits_{j=2}^{n} P_1P_j$ is the length of the star rooted at P_1, the point of $\mathcal{P}$ that is closest to the Fermat-Torricelli center O. Obviously,

$$\alpha(\mathcal{P}) \le \min\{\beta(\mathcal{P}), \gamma(\mathcal{P})\} \tag{6}$$

as $S_{min}(\mathcal{P}) \le S_1(\mathcal{P})$, and $S_{min}(\mathcal{P}) \le S_{avg}(\mathcal{P})$.

All known upper estimates for $\alpha(\mathcal{P})$ were obtained by first bounding $\beta(\mathcal{P})$ and $\gamma(\mathcal{P})$, and then using inequality (6). The next lemma provides lower and upper bounds for the length of the rooted stars if the points are restricted to those that lie on the unit circle centered at O.

Lemma 3.3. [7] *Let $Q_1, Q_2, \ldots, Q_n$ be points on the unit circle centered at the origin O, such that $\overrightarrow{OQ_1} + \overrightarrow{OQ_2} + \ldots + \overrightarrow{OQ_n} = \overrightarrow{0}$. Then for every $1 \le i \le n$,*

$$n \le \sum_{j=1}^{n} Q_iQ_j \le n\sqrt{2}. \tag{7}$$

Proof. Let us first note that

$$\sum_{j=1}^{n} Q_iQ_j^2 = \sum_{j=1}^{n} \|\overrightarrow{OQ_i} - \overrightarrow{OQ_j}\|^2 = \sum_{j=1}^{n}(OQ_i^2 + OQ_j^2 - 2\overrightarrow{OQ_i}\cdot\overrightarrow{OQ_j}) =$$

$$= 2n - 2\overrightarrow{OQ_i}\cdot\sum_{j=1}^{n}\overrightarrow{OQ_j} = 2n. \tag{8}$$

Using equality (8) and the fact that $Q_iQ_j \ge Q_iQ_j^2/2$, we immediately obtain that

$$\sum_{j=1}^{n} Q_iQ_j \ge \frac{1}{2}\sum_{j=1}^{n} Q_iQ_j^2 = \frac{1}{2}\cdot 2n = n, \quad \text{which proves the lower bound.}$$

Using (8) again and the Cauchy-Schwarz inequality, we have that

$$\left(\sum_{j=1}^{n} Q_iQ_j\right)^2 \le n\sum_{j=1}^{n} Q_iQ_j^2 = 2n^2, \quad \text{and the upper bound follows.}$$

□

We now have all the ingredients necessary to present a proof of Fekete's and Meijer's bound for $\beta(\mathcal{P})$.

Theorem 3.1. [7] *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane. Suppose that O is the Fermat-Torricelli center of $\mathcal{P}$, and let $OP_i = 1 + r_i$, for $1 \leq i \leq n$, with $0 = r_1 \leq r_2 \leq \ldots r_n$.*

$$\alpha(\mathcal{P}) \leq \beta(\mathcal{P}) \leq \frac{\sqrt{2}+R}{1+R} \leq \sqrt{2}, \quad \text{where } nR = \sum_{i=1}^{n} r_i. \tag{9}$$

Proof. We find an upper estimate for $S_1(\mathcal{P})$ by using triangle inequality and the upper bound in (7):

$$S_1(\mathcal{P}) = \sum_{j=1}^{n} P_1P_j \leq \sum_{j=1}^{n}(Q_1Q_j + r_j) = \sum_{j=1}^{n} Q_1Q_j + nR \leq n(\sqrt{2}+R)$$

On the other hand, by (2) we know that $FT(\mathcal{P}) = n(1+R)$. Using (5) the result follows. □

We move on now to bounding $\gamma(\mathcal{P})$. We introduce first a useful

Notation. *Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a finite point set in the plane. Then $D(\mathcal{A})$ denotes the sum of all interpoint distances determined by pairs of points in $\mathcal{C}$.*

$$D(\mathcal{C}) = D(\{C_1, C_2, \ldots, C_m\}) = \sum_{1 \leq i < j \leq m} C_iC_j. \tag{10}$$

With this notation, the expression of $\gamma(\mathcal{P})$ defined in (5) becomes

$$\gamma(\mathcal{P}) = \frac{2D(\mathcal{P})}{|\mathcal{P}|\, FT(\mathcal{P})}. \tag{11}$$

The following result of Fejes Tóth provides a tight upper bound for the sum of interpoint distances if the point set is restricted to a circle.

Lemma 3.4. [6] *Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a set of points lying on a unit circle. Then*

$$D(\mathcal{C}) \leq m \cot\left(\frac{\pi}{2m}\right) \leq cm^2 = c|\mathcal{C}|^2, \quad \text{where} \quad c := 2/\pi. \tag{12}$$

The first inequality becomes equality if and only if $\mathcal{B}$ is the vertex set of a regular m-gon. The bound (12) is essential in obtaining various estimates later on. To simplify the calculations, we will consistently use c instead of $2/\pi$. The next lemma gives bounds for the distances P_iP_j, $1 \leq i < j \leq n$.

Lemma 3.5. [9] *Let P_i, P_j be two distinct points such that $OP_i = 1 + r_i$, $OP_j = 1 + r_j$, with $0 \leq r_i \leq r_j$. Let Q_i, Q_j be the points where the unit circle centered at O intersect the segments OP_i and OP_j, respectively. Then*

$$\frac{1}{2}Q_iQ_j^2(1+r_i) + r_j - r_i \leq P_iP_j \leq Q_iQ_j(1+r_i) + r_j - r_i \leq Q_iQ_j + r_i + r_j. \tag{13}$$

Due to space restrictions we omit the proofs of these two lemmata. We will next present the upper bound for $\gamma(\mathcal{P})$ due to Dumitrescu, Tóth and Xu.

Theorem 3.2. [5] *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane. Suppose that O is the Fermat-Torricelli center of $\mathcal{P}$ and let $OP_i = 1 + r_i$, for $1 \leq i \leq n$, with $0 = r_1 \leq r_2 \leq \ldots r_n$.*

$$\alpha(\mathcal{P}) \leq \gamma(\mathcal{P}) \leq \frac{2c+2R}{1+R}, \quad \text{where } nR = \sum_{i=1}^{n} r_i. \tag{14}$$

Proof. Summing $P_iP_j \le Q_iQ_j + r_i + r_j$ over all $1 \le i < j \le n$, we obtain

$$D(\mathcal{P}) \le \sum_{1\le i<j\le n} (Q_iQ_j + r_i + r_j) = D(\{Q_1, Q_2, \dots, Q_n\}) + (n-1)nR \le$$
$$\le cn^2 + (n-1)nR < n^2(c+R),$$

where we used (12) for the set $\{Q_1, Q_2, \dots, Q_n\}$.
If follows that $D(\mathcal{P}) = \sum_{1\le i<j\le n} P_iP_j \le n^2(c+R)$, and after using this inequality into (11)

$$\gamma(\mathcal{P}) = \frac{2D(\mathcal{P})}{|\mathcal{P}|\, FT(\mathcal{P})} \le \frac{2n^2(c+R)}{n^2(1+R)} = \frac{2c+2R}{1+R}, \quad \text{as claimed.}$$

□

Now combining (6), (9) and (14) we obtain that

$$\alpha(\mathcal{P}) \le \min\{\beta(\mathcal{P}), \gamma(\mathcal{P})\} \le \min\left\{\frac{\sqrt{2}+R}{1+R}, \frac{2c+2R}{1+R}\right\} \le 1.3630\dots, \tag{15}$$

with equality when $R = \sqrt{2} - 2c = \sqrt{2} - 4/\pi = 0.1409\dots$. This is the bound Dumitrescu, Tóth and Xu published in [5]. Ismailescu and Park [9] improved this bound by using the better estimate $P_iP_j \le Q_iQ_j(1+r_i) + r_j - r_i$ rather than $P_iP_j \le Q_iQ_j + r_i + r_j$. They needed the following additional result, which gives an upper bound for r_n in terms of R.

Lemma 3.6. [9] *Let $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ be an optimal (in the sense of Lemma 3.1) set of n points in the plane. Suppose that O is the Fermat-Torricelli center of $\mathcal{P}$, and let $OP_i = 1 + r_i$, for $1 \le i \le n$, with $0 = r_1 \le r_2 \le \ldots \le r_n$. Then*

$$r_n \le \sqrt{2} - 1 + 2R. \tag{16}$$

Proof. By Lemma 3.2, all rooted stars of $\mathcal{P}$ will be of the same length. In particular, $S_1(\mathcal{P}) = S_n(\mathcal{P})$. On one hand, using the upper bounds from both (7) and (13) we obtain that

$$S_1(\mathcal{P}) = \sum_{j=1}^{n} P_1P_j \le \sum_{j=1}^{n}(Q_1Q_j + r_j) = \sum_{j=1}^{n} Q_1Q_j + \sum_{j=1}^{n} r_j \le n\sqrt{2} + nR = n(\sqrt{2}+R).$$

On the other hand, using the lower bounds from both (7) and (13), as well as (8), we get

$$S_n(\mathcal{P}) = \sum_{i=1}^{n} P_iP_n \ge \sum_{i=1}^{n}\left(\frac{1}{2}Q_iQ_n^2 + r_n - r_i\right) =$$
$$= \frac{1}{2}\sum_{i=1}^{n} Q_iQ_n^2 + \sum_{i=1}^{n}(r_n - r_i) = n(1 + r_n - R).$$

Using the last two inequalities we obtain that

$$n(1 + r_n - R) \le S_n(\mathcal{P}) = S_1(\mathcal{P}) \le n(\sqrt{2} + R), \quad \text{from which} \quad r_n \le \sqrt{2} - 1 + 2R,$$

as claimed. □

4. A better way to bound $\gamma(\mathcal{P})$

The weak point of both [5] and [9] is that the upper bounds for P_iP_j listed in (13) are rather crude. Even the better of these two estimates is tight only when the angle between the rays $\overrightarrow{OP_i}$ and $\overrightarrow{OP_j}$ is either 0 or π. Recall that these bounds were needed in order to find an upper bound for $D(\mathcal{P})$, and eventually $\gamma(\mathcal{P})$. In this section we present a more efficient way to deal with this issue.

The key is to consider the set $\mathcal{P}' = \{P_1', P_2', \ldots, P_n'\}$, the reflection of $\mathcal{P}$ across the Fermat-Torricelli center, O. Here, $P_k' = -P_k$, the symmetrical of P_k with respect to O.

Consider the $2n$ point collection $\mathcal{P} \cup \mathcal{P}'$. It may be that some point appears in both $\mathcal{P}$ and $\mathcal{P}'$; nevertheless, if that is the case, we count that point once for each of the two sets.

The following result is crucial for our purposes.

Theorem 4.1. *([11, 12])*

Let $A_1, A_2, \ldots, A_n$, $A_1', A_2', \ldots, A_n'$ be two collections of (not necessarily distinct) points in the plane. Then,

$$\sum_{1 \le i < j \le n} A_iA_j + \sum_{1 \le i < j \le n} A_i'A_j' \le \sum_{i=1}^{n} \sum_{j=1}^{n} A_iA_j'. \tag{17}$$

Proof. (Sketch)

A metric space (X, d) is said to be of *negative-type* if for every finite choice of points $x_1, x_2, \ldots x_m$ in X and every set of real numbers $\alpha_1, \alpha_2, \ldots \alpha_m$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 0$ we have

$$\sum_{i,j=1}^{m} \alpha_i \alpha_j \, d(x_i, x_j) \le 0. \tag{18}$$

Schoenberg [12] proved that $\mathbb{R}^d$ with the usual Euclidean distance is of negative-type.

Substituting $m = 2n$, $\{x_1, x_2, \ldots x_m\} = \{A_1, A_2, \ldots, A_n, A_1', A_2', \ldots A_n'\}$ and $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1$, $\alpha_1' = \alpha_2' = \ldots = \alpha_n' = -1$ into inequality (18), confirms inequality (17).

A short direct proof of the same result was given by Morgenstern [11]. □

We can then easily prove the following:

Theorem 4.2. *For every set $\mathcal{P}$ we have*

$$\gamma(\mathcal{P} \cup \mathcal{P}') \ge \gamma(\mathcal{P}), \tag{19}$$

where $\gamma(.)$ is defined in (11).

Proof. It is immediate that $\mathcal{P}$ and $\mathcal{P} \cup \mathcal{P}'$ have the same Fermat-Torricelli center. Moreover,

$$FT(\mathcal{P} \cup \mathcal{P}') = 2FT(\mathcal{P}).$$

On the other hand, by using Theorem 4.1 we have

$$D(\mathcal{P} \cup \mathcal{P}') = D(\mathcal{P}) + D(\mathcal{P}') + \sum_{A \in \mathcal{P}, B \in \mathcal{P}'} AB \ge 2(D(\mathcal{P}) + D(\mathcal{P}')) = 4D(\mathcal{P}).$$

Putting things together,

$$\gamma(\mathcal{P}\cup\mathcal{P}') = \frac{2D(\mathcal{P}\cup\mathcal{P}')}{|\mathcal{P}\cup\mathcal{P}'|\cdot FT(\mathcal{P}\cup\mathcal{P}')} \geq \frac{8D(\mathcal{P})}{2|\mathcal{P}|\cdot 2FT(\mathcal{P})} = \frac{2D(\mathcal{P})}{|\mathcal{P}|\cdot FT(\mathcal{P})} = \gamma(\mathcal{P}).$$

□

The relevance of Theorem 4.2 should be clear: any upper bound for $\gamma(\mathcal{P}\cup\mathcal{P}')$ is also an upper bound for $\gamma(\mathcal{P})$. On the other hand, one may have an easier time exploiting the symmetry of $\mathcal{P}\cup\mathcal{P}'$. This is indeed the case.

Refer back to Theorem 2.1. We plan to show that $\alpha(\mathcal{P}) \leq 1.3385$.

By Fekete's and Meijer's Theorem 3.1, we already know that $\alpha(\mathcal{P}) \leq (\sqrt{2}+R)/(1+R)$. The right-hand side is a decreasing function of R, and if $R \geq 15/16-\sqrt{2}/2 = 0.2303\ldots$ then already $\alpha(\mathcal{P}) \leq 1.3366\ldots$, and there is nothing left to prove.

So, for the rest of the paper, one can safely assume that

$$R \leq \frac{15}{16} - \frac{\sqrt{2}}{2} = 0.2303\ldots. \tag{20}$$

Next, using (16) we obtain an upper bound for r_n:

$$r_n \leq \sqrt{2} - 1 + 2R \leq \frac{7}{8}. \tag{21}$$

We desire to find an upper bound for $D(\mathcal{P}\cup\mathcal{P}')$. This will eventually lead to an upper bound for $\alpha(\mathcal{P})$ because

$$\gamma(\mathcal{P}\cup\mathcal{P}') \geq \gamma(\mathcal{P}) \geq \alpha(\mathcal{P}) \tag{22}$$

The trick in bounding $D(\mathcal{P}\cup\mathcal{P}')$ is to estimate *four* segments at a time, rather than just one. The details are presented in the next two lemmata.

Here and in the sequel, $Per(ABCD)$ stands for the perimeter of the parallelogram $ABCD$.

Lemma 4.1. *Let $P_iP_jP_i'P_j'$ be a parallelogram centered at O. Assume that $OP_i = OP_{i'} = 1+r_i$, $OP_j = OP_{j'} = 1+r_j$, where $0 \leq r_i \leq r_j \leq 7/8$. Let Q_i be the point where the segment OP_i intersects the circle of unit radius centered at O. The points Q_j, Q_i', and Q_j' are defined analogously. Then*

(23)

$$Per(P_iP_jP_i'P_j') \leq (1+r_i)\,Per(Q_iQ_jQ_i'Q_j')+4(r_j-r_i)-\lambda\,(r_j-r_i)(Per(Q_iQ_jQ_i'Q_j')-4),$$

where $\lambda = (6-2\sqrt{2})/7 = 0.453\ldots$.

Before we present the proof, we make one observation. As mentioned earlier, in [9] it is proved that

$$P_iP_j \leq (1+r_i)Q_iQ_j + (r_j - r_i).$$

Using the same inequality for bounding P_jP_i', $P_i'P_j'$ and $P_j'P_i$, it follows immediately that

$$Per(P_iP_jP_i'P_j') \leq (1+r_i)\cdot Per(Q_iQ_jQ_i'Q_j') + 4(r_j - r_i).$$

This estimate eventually leads to an upper bound of $1.3545\ldots$ for $\alpha(\mathcal{P})$. It is clear that inequality (23) is stronger since $r_j - r_i \geq 0$, and $Per(Q_iQ_jQ_i'Q_j') \geq 4$, as every rectangle inscribed in a unit circle has perimeter at least 4. It is exactly this last term that will allow us to prove a better bound for $\alpha(\mathcal{P})$.

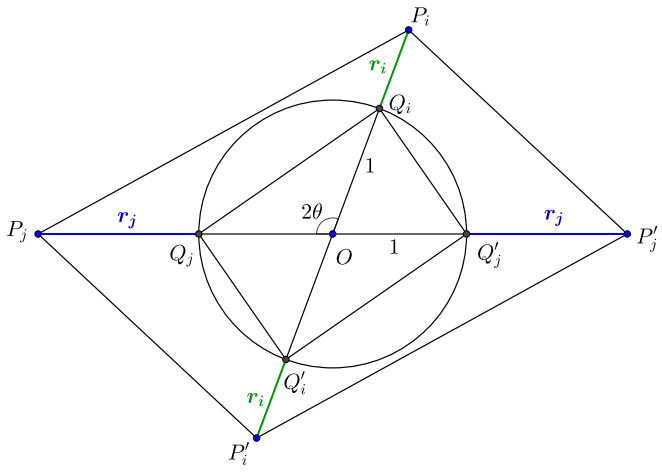

FIGURE 2. An upper bound for the perimeter of the parallelogram $P_iP_jP'_iP'_j$

Proof. Denote $\angle P_iOP_j = 2\theta$, where $0 \leq \theta \leq \pi/2$. Using the law of cosines, it is easy to show that

$$Q_iQ_j = Q'_iQ'_j = 2\sin\theta, \qquad P_iP_j = P'_iP'_j = \sqrt{(r_j - r_i)^2 + 4\sin^2\theta(1+r_i)(1+r_j)},$$
$$Q_jQ'_i = Q'_jQ_i = 2\cos\theta, \qquad P_jP'_i = P'_jP_i = \sqrt{(r_j - r_i)^2 + 4\cos^2\theta(1+r_i)(1+r_j)}.$$

As a result, $Per(Q_iQ_jQ'_iQ'_j) = 4(\sin\theta + \cos\theta)$, and

$$Per(P_iP_jP'_iP'_j) = 2\sqrt{(r_j - r_i)^2 + 4\sin^2\theta(1+r_i)(1+r_j)} + \\ + 2\sqrt{(r_j - r_i)^2 + 4\cos^2\theta(1+r_i)(1+r_j)}.$$

After dividing both sides by 2, (23) reduces to

$$\sqrt{(r_j - r_i)^2 + 4\sin^2\theta(1+r_i)(1+r_j)} + \sqrt{(r_j - r_i)^2 + 4\cos^2\theta(1+r_i)(1+r_j)} \leq \\ \leq 4(1+r_i)(\sin\theta + \cos\theta) + 4(r_j - r_i) - 4\lambda(r_j - r_i)(\sin\theta + \cos\theta - 1).$$

It is easy to see that if $r_j = r_i$ then equality holds. We will thus assume that $r_j > r_i$. Denote $y := 1 + r_i$ and $z := r_j - r_i > 0$. It follows that $1 + r_j = y + z$ and the above inequality becomes

$$\sqrt{z^2 + 4y(y+z)\sin^2\theta} + \sqrt{z^2 + 4y(y+z)\cos^2\theta} \leq 2(y - \lambda z)(\sin\theta + \cos\theta) + 2(1+\lambda)z,$$

which after dividing by z and rearranging can be written as

$$\sqrt{\left(2\frac{y}{z} + 1\right)^2 \sin^2\theta + \cos^2\theta} + \sqrt{\left(2\frac{y}{z} + 1\right)^2 \cos^2\theta + \sin^2\theta} \leq \\ \leq \left(2\frac{y}{z} + 1\right)(\sin\theta + \cos\theta) - (1 + 2\lambda)(\sin\theta + \cos\theta) + 2(1+\lambda).$$

Finally, let

$$u := 2\frac{y}{z} + 1 = \frac{2 + r_i + r_j}{r_j - r_i}.$$

Since $0 \leq r_i < r_j \leq 7/8$ it immediately follows that $u \geq 23/7$.

It is therefore enough to prove that for $u \geq 23/7$, $0 \leq \theta \leq \pi/2$, and $\lambda = (6 - 2\sqrt{2})/7$ we have that

$$\sqrt{u^2 \sin^2\theta + \cos^2\theta} + \sqrt{u^2 \cos^2\theta + \sin^2\theta} \leq (u - 1 - 2\lambda)(\sin\theta + \cos\theta) + 2(1 + \lambda).$$

A proof of this inequality is presented in the Appendix - see Fact 6.1. □

Lemma 4.2. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane, and let O be the Fermat-Torricelli point of $\mathcal{P}$. Assume that $\min OP_i = 1$, and that the points are numbered such that $OP_i \leq OP_j$, for all $1 \leq i < j \leq n$. For each $1 \leq i \leq n$, denote $OP_i = 1 + r_i$; by the previous assumption, it follows that $0 = r_1 \leq r_2 \leq r_3 \leq \ldots r_{n-1} \leq r_n$. Further, assume that $r_n \leq 7/8$. Use the following notations:*

$$R = \frac{r_1 + r_2 + \ldots + r_n}{n}, \quad c = \frac{2}{\pi}, \quad \lambda = \frac{6 - 2\sqrt{2}}{7}.$$

Let $\mathcal{P}' = \{P_1', P_2', \ldots, P_n'\}$ be the reflection of $\mathcal{P}$ with respect to point O; that is, $P_i' = -P_i$, for all $1 \leq i \leq n$. Then the sum of all the distances determined by the $2n$ points in $\mathcal{P} \cup \mathcal{P}'$ is bounded by

$$D(\mathcal{P} \cup \mathcal{P}') \leq 4n^2 \left(c + (1 + \lambda - 2c\lambda)\, R - \frac{8}{7}(1 + \lambda - c - 2c\,\lambda) R^2 \right). \tag{24}$$

Proof. We split the set of $\binom{2n}{2}$ distances determined by the points of $\mathcal{P} \cup \mathcal{P}'$ into two classes. In the first class, we consider the n segments of type P_iP_i', $1 \leq i \leq n$. The sum of the remaining $2n^2 - 2n$ distances is the same as the sum of the perimeters of the parallelograms $P_iP_jP_i'P_j'$, where $1 \leq i < j \leq n$. There are $\binom{n}{2}$ such parallelograms, and each contributes four distances to the sum, for a total of $4 \cdot \binom{n}{2} = 2n^2 - 2n$ distances. Hence,

$$D(\mathcal{P} \cup \mathcal{P}') = \sum_{i=1}^{n} P_iP_i' + \sum_{1 \leq i < j \leq n} Per(P_iP_jP_i'P_j'). \tag{25}$$

The first sum is very easy to calculate

$$\sum_{i=1}^{n} P_iP_i' = \sum_{i=1}^{n} (2 + 2r_i) = 2n + 2nR. \tag{26}$$

For the second sum we will use the bound from Lemma 4.1. After rearranging the terms and summing over all $1 \leq i < j \leq n$ we have

$$\begin{aligned} \sum_{1 \leq i < j \leq n} Per(P_iP_jP_i'P_j') \leq & \sum_{1 \leq i < j \leq n} Per(Q_iQ_jQ_i'Q_j') + \sum_{1 \leq i < j \leq n} r_i\, Per(Q_iQ_jQ_i'Q_j') + \\ & + (4 + 4\lambda) \sum_{1 \leq i < j \leq n} (r_j - r_i) - \lambda \sum_{1 \leq i < j \leq n} (r_j - r_i) Per(Q_iQ_jQ_i'Q_j'). \end{aligned} \tag{27}$$

We will compute each of the four sums that appear in (27) separately, and then we will combine the results. The following identities are easy to establish and will

be used in the sequel.

$$\sum_{1\le i<j\le n}\left(x_{ij}\sum_{k=1}^{i-1}y_k\right)=\sum_{k=1}^{n-1}\left(y_k\sum_{k+1\le i<j\le n}x_{ij}\right). \tag{28}$$

$$\sum_{1\le i<j\le n}\left(x_{ij}\sum_{k=i}^{j-1}y_k\right)=\sum_{k=1}^{n-1}\left(y_k\sum_{i=1}^{k}\sum_{j=k+1}^{n}x_{ij}\right). \tag{29}$$

$$\sum_{k=1}^{n-1}x_k(y_{k+1}-y_k)=x_ny_n-x_0y_1+\sum_{k=1}^{n}(x_{k-1}-x_k)y_k. \tag{30}$$

We introduce two more notations. For every $0\le k\le n$, define

$$A_k=D(\{Q_1,Q_1',Q_2,Q_2',\dots,Q_k,Q_k'\}), \tag{31}$$

$$B_k=D(\{Q_{k+1},Q_{k+1}',Q_{k+2},Q_{k+2}',\dots,Q_n,Q_n'\}). \tag{32}$$

Note that $A_n=B_0$, while trivially, $A_0=B_n=0$. More importantly, by Fejes Tóth's Theorem 3.4

$$A_k\le 4ck^2 \quad\text{and}\quad B_k\le 4c(n-k)^2. \tag{33}$$

For the first sum in (27) we have

(34)

$$\sum_{1\le i<j\le n}Per(Q_iQ_jQ_i'Q_j')=D(\{Q_1,Q_1',Q_2,Q_2',\dots,Q_n,Q_n'\})-\sum_{i=1}^{n}Q_iQ_i'=B_0-2n.$$

The second sum in (27) can be written as

$$\begin{aligned}&\sum_{1\le i<j\le n}r_i\,Per(Q_iQ_jQ_i'Q_j')=\sum_{1\le i<j\le n}\left(Per(Q_iQ_jQ_i'Q_j')\sum_{k=1}^{i-1}(r_{k+1}-r_k)\right)=\\&=\sum_{k=1}^{n-1}\left((r_{k+1}-r_k)\sum_{k+1\le i<j\le n}Per(Q_iQ_jQ_i'Q_j')\right)=\\&=\sum_{k=1}^{n-1}(r_{k+1}-r_k)\left(D(\{Q_{k+1},Q_{k+1}',\dots,Q_n,Q_n'\})-\sum_{u=k+1}^{n}Q_uQ_u'\right)=\\&=\sum_{k=1}^{n-1}(r_{k+1}-r_k)(B_k-2(n-k))=\sum_{k=1}^{n-1}(r_{k+1}-r_k)(B_k-2n+2k).\end{aligned} \tag{35}$$

The third sum in (27) is easy to calculate:

$$\begin{aligned}\sum_{1\le i<j\le n}(r_j-r_i)&=\sum_{1\le i<j\le n}\left(\sum_{k=i}^{j-1}(r_{k+1}-r_k)\right)=\sum_{1\le i<j\le n}\left(1\cdot\sum_{k=i}^{j-1}(r_{k+1}-r_k)\right)=\\&=\sum_{k=1}^{n-1}(r_{k+1}-r_k)\left(\sum_{i=1}^{k}\sum_{j=k+1}^{n}1\right)=\sum_{k=1}^{n-1}(r_{k+1}-r_k)k(n-k).\end{aligned} \tag{36}$$

Finally, we express the last sum in (27) as follows:

$$\sum_{1\le i<j\le n}(r_j-r_i)\,Per(Q_iQ_jQ_i'Q_j') = \sum_{1\le i<j\le n}\left(Per(Q_iQ_jQ_i'Q_j')\sum_{k=i}^{j-1}(r_{k+1}-r_k)\right) = \\ = \sum_{k=1}^{n-1}\left((r_{k+1}-r_k)\sum_{i=1}^{k}\sum_{j=k+1}^{n}Per(Q_iQ_jQ_i'Q_j')\right).$$

But it is easy to see that the inner double sum can be written as

$$\sum_{i=1}^{k}\sum_{j=k+1}^{n}Per(Q_iQ_jQ_i'Q_j') = D(\{Q_1,Q_1',Q_2,Q_2',\dots,Q_n,Q_n'\})- \\ D(\{Q_1,Q_1',Q_2,Q_2',\dots,Q_k,Q_k'\}) - D(\{Q_{k+1},Q_{k+1}',\dots,Q_n,Q_n'\}) = B_0 - A_k - B_k.$$

It follows that

$$\sum_{1\le i<j\le n}(r_j-r_i)\,Per(Q_iQ_jQ_i'Q_j') = \sum_{k=1}^{n-1}(r_{k+1}-r_k)(B_0-A_k-B_k). \tag{37}$$

Combining now (26), (34), (35), (36) and (37), relation (25) gives

$$D(\mathcal{P}\cup\mathcal{P}') \le 2nR + B_0 + \sum_{k=1}^{n-1}(r_{k+1}-r_k)(B_k-2n+2k)+ \\ + (4+4\lambda)\sum_{k=1}^{n-1}(r_{k+1}-r_k)(nk-k^2) - \lambda\sum_{k=1}^{n-1}(r_{k+1}-r_k)(B_0-A_k-B_k),$$

which, after taking into account that $\sum_{k=1}^{n-1}(r_{k+1}-r_k)B_0 = r_n\,B_0$, can be written as

$$D(\mathcal{P}\cup\mathcal{P}') \le 2nR + B_0(1-\lambda r_n)+ \\ + \sum_{k=1}^{n-1}(r_{k+1}-r_k)\left(\lambda A_k + (1+\lambda)B_k + (4+4\lambda)(nk-k^2) - 2n + 2k\right).$$

Using now the bounds for A_k and B_k mentioned in (33) and the fact that $r_{k+1}-r_k \ge 0$, for all $1\le k\le n-1$, we obtain that

$$D(\mathcal{P}\cup\mathcal{P}') \le 2nR + 4c(1-\lambda r_n)n^2+ \\ + \sum_{k=1}^{n-1}(r_{k+1}-r_k)\left(4c\lambda k^2 + 4c(1+\lambda)(n-k)^2 + (4+4\lambda)(nk-k^2) - 2n + 2k\right),$$

which after rearranging the terms becomes

$$D(\mathcal{P}\cup\mathcal{P}') \le 2nR + 4c(1-\lambda r_n)n^2 + 4c(1+\lambda)n^2\sum_{k=1}^{n-1}(r_{k+1}-r_k)+ \\ + \sum_{k=1}^{n-1}(-2n+2k)(r_{k+1}-r_k) + 4(c+2c\lambda-1-\lambda)\sum_{k=1}^{n-1}k^2(r_{k+1}-r_k)+ \\ + 4(1-2c)(1+\lambda)n\sum_{k=1}^{n-1}k(r_{k+1}-r_k). \tag{38}$$

Using identity (30) with appropriate choices for x_k and y_k, one can easily show that

$$\sum_{k=1}^{n-1}(r_{k+1}-r_k)=r_n, \tag{39}$$

$$\sum_{k=1}^{n-1}k(r_{k+1}-r_k)=nr_n-nR, \tag{40}$$

$$\sum_{k=1}^{n-1}k^2(r_{k+1}-r_k)=n^2r_n+nR-\sum_{k=1}^{n-1}2k\,r_k, \quad \text{and} \tag{41}$$

$$\sum_{k=1}^{n-1}(-2n+2k)(r_{k+1}-r_k)=-2nR. \tag{42}$$

Notice that three of the four sums above depend only on n, R, and r_n.

To deal with the remaining sum we need the following:

Fact 4.1. [9] *Suppose that* $0=r_1\le r_2\le \dots r_{n-1}\le r_n\le 7/8$ *is a nondecreasing sequence of nonnegative numbers and* $\sum_{k=1}^{n} r_k=nR$ *is fixed. Then*

$$\sum_{k=1}^{n}2kr_k\le n^2\left(2R-\frac{8}{7}\,R^2\right)+nR. \tag{43}$$

We present a proof of Fact 4.1 in the Appendix. Using now inequality (43) in (41), we obtain

$$\sum_{k=1}^{n-1}k^2(r_{k+1}-r_k)=n^2r_n+nR-\sum_{k=1}^{n-1}2k\,r_k\ge n^2\left(r_n-2R+\frac{8}{7}\,R^2\right).$$

Recall that $c=2/\pi$ and $\lambda=(6-2\sqrt{2})/7$. It follows that the coefficient of $\sum_{k=1}^{n-1}k^2(r_{k+1}-r_k)$ in (38) is $4(c+2c\lambda-1-\lambda)=-0.958\ldots<0$. This implies that

$$4(c+2c\lambda-1-\lambda)\sum_{k=1}^{n-1}k^2(r_{k+1}-r_k)\le 4n^2(c+2c\lambda-1-\lambda)\left(r_n-2R+\frac{8}{7}\,R^2\right). \tag{44}$$

We now combine (39), (40), (42) and (44) into (38) to obtain

$$D(\mathcal{P}\cup\mathcal{P}')\le 2nR+4c(1-\lambda r_n)n^2+4c(1+\lambda)n^2r_n+4(1-2c)(1+\lambda)n^2(r_n-R)+$$
$$+4(c+2c\lambda-1-\lambda)\left(r_n-2R+\frac{8}{7}\,R^2\right)-2nR,$$

which after simplifying gives

$$D(\mathcal{P}\cup\mathcal{P}')\le 4n^2\left(c+(1+\lambda-2c\lambda)\,R-\frac{8}{7}(1+\lambda-c-2c\,\lambda)R^2\right), \text{as claimed in (24).}$$

The proof of Lemma 4.2 is now complete. □

We are now ready to prove Theorem 2.1.

Main result. *Let $c = 2/\pi$, $\lambda = (6-2\sqrt{2})/7$. For every point set $\mathcal{P}$ in the plane we have*

(45)

$$\alpha(\mathcal{P}) \le \min\left\{\frac{\sqrt{2}+R}{1+R}, \frac{2c+(2+2\lambda-4c\lambda)R-16(1+\lambda-c-2c\lambda)R^2/7}{1+R}\right\} \le 1.33845\ldots.$$

Proof. The first part is a restatement of Theorem 3.1. For the second part we use Lemma 4.2 and inequality (22).

$$\alpha(\mathcal{P}) \le \gamma(\mathcal{P}) \le \gamma(\mathcal{P}\cup\mathcal{P}') = \frac{2D(\mathcal{P}\cup\mathcal{P}')}{|\mathcal{P}\cup\mathcal{P}'|\cdot FT(\mathcal{P}\cup\mathcal{P}')} = \frac{2D(\mathcal{P}\cup\mathcal{P}')}{2|\mathcal{P}|\cdot 2FT(\mathcal{P})} \le$$
$$\le \frac{4n^2\left(c+(1+\lambda-2c\lambda)R-\frac{8}{7}(1+\lambda-c-2c\,\lambda)R^2\right)}{2n^2(1+R)} =$$
$$= \frac{2c+(2+2\lambda-4c\lambda)R-\frac{16}{7}(1+\lambda-c-2c\lambda)R^2}{1+R}.$$

The two functions on the right-hand side become equal when $R = 0.2238\ldots$. □

The graph below shows the known bounds for $\alpha(\mathcal{P})$.

- $\alpha(\mathcal{P}) \le \beta(\mathcal{P}) \le \dfrac{\sqrt{2}+R}{1+R}$, Fekeke and Meijer.
- $\alpha(\mathcal{P}) \le \gamma(\mathcal{P}) \le \dfrac{\sqrt{2c}+2R}{1+R}$, Dumitrescu, Tóth and Xu.
- $\alpha(\mathcal{P}) \le \gamma(\mathcal{P}) \le \dfrac{2c+2R-\frac{16}{7}(1-c)R^2}{1+R}$, Ismailescu and Park.
- $\alpha(\mathcal{P}) \le \gamma(\mathcal{P}) \le \dfrac{2c+2R-(4c-2)\lambda R-\frac{16}{7}(1+\lambda-c-2c\lambda)R^2}{1+R}$, our bound.

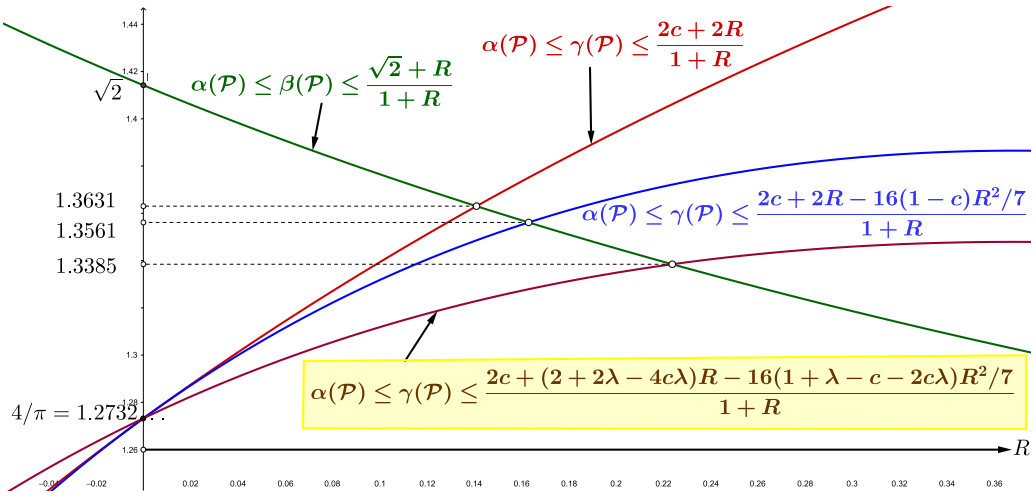

FIGURE 3. Gradual improvements in estimating $\alpha(n)$

5. The squared Euclidean distance variation

What if rather than using the standard Euclidean distance we employ the squared Euclidean distance instead? Then the problem becomes much simpler, as we will show below.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane. For every $1 \leq i \leq n$ define

$$SS_i(\mathcal{P}) = \sum_{j=1}^{n} P_iP_j^2, \quad \text{and} \quad SS_{min}(\mathcal{P}) = \min_{1\leq i\leq n} SS_i(\mathcal{P}). \tag{46}$$

Let O be the point for which the sum $\sum_{i=1}^{n} OP_i^2$ takes the minimum value. It is easy to show that O is the centroid of $\mathcal{P}$. Unlike the Fermat-Torricelli center, the location of the centroid can be easily expressed, because it is the arithmetic mean of the points in $\mathcal{P}$. Denote

$$\Gamma(\mathcal{P}) = \sum_{i=1}^{n} OP_i^2. \tag{47}$$

Theorem 5.1. *With the notations above we have*

$$\frac{SS_{min}(\mathcal{P})}{\Gamma(\mathcal{P})} \leq 2, \tag{48}$$

with equality if and only if all points of $\mathcal{P}$ lie on a circle centered at the centroid of $\mathcal{P}$.

Proof. Without loss of generality assume that the centroid is located at the origin. For each $1 \leq i \leq n$, denote $\overrightarrow{OP_i} = \mathbf{v}_i$. Then

$$SS_i(\mathcal{P}) = \sum_{j=1}^{n} \|\mathbf{v}_i - \mathbf{v}_j\|^2 = \sum_{j=1}^{n} \left(\|\mathbf{v}_i\|^2 + \|\mathbf{v}_j\|^2 - 2\mathbf{v}_i \cdot \mathbf{v}_j\right) =$$

$$= n\|\mathbf{v}_i\|^2 + \sum_{j=1}^{n} \|\mathbf{v}_j\|^2 - 2\mathbf{v}_i \cdot \sum_{j=1}^{n} \mathbf{v}_j = n\|\mathbf{v}_i\|^2 + \Gamma(\mathcal{P}). \tag{49}$$

Summing the identity above over all $1 \leq i \leq n$ we obtain that

$$SS_{min}(\mathcal{P}) \leq \frac{1}{n}\sum_{i=1}^{n} SS_i(\mathcal{P}) = \frac{1}{n}\sum_{i=1}^{n} \left(n\|\mathbf{v}_i\|^2 + \Gamma(\mathcal{P})\right) = 2\Gamma(\mathcal{P}).$$

This proves inequality (48). Equality happens only when $SS_{min}(\mathcal{P}) = \frac{1}{n}\sum_{i=1}^{n} SS_i(\mathcal{P})$; that is, when $SS_i(\mathcal{P}) = SS_j(\mathcal{P})$ for all $1 \leq i < j \leq n$. If we use (49), it immediately follows that $\|\mathbf{v}_i\| = \|\mathbf{v}_j\|$, for all $i \neq j$; that is, the points of $\mathcal{P}$ are on a circle centered at the origin. □

6. Appendix

Fact 6.1. *Consider the function*

$$f(u,\theta) = (u-1-2\lambda)(\sin\theta+\cos\theta)+2(1+\lambda)-\sqrt{u^2\sin^2\theta+\cos^2\theta}-\sqrt{u^2\cos^2\theta+\sin^2\theta}.$$

where $u \geq 23/7, 0 \leq \theta \leq \pi/2$, and $\lambda = (6-2\sqrt{2})/7$.

Then $f(u,\theta) \geq 0$ with equality if and only if $\theta \in \{0, \pi/2\}$ or $(u,\theta) = (23/7, \pi/4)$.

Proof. Start by computing the partial derivative of f with respect to u

$$\frac{\partial f}{\partial u} = \left(\sin\theta - \frac{u\sin^2\theta}{\sqrt{u^2\sin^2\theta+\cos^2\theta}}\right) + \left(\cos\theta - \frac{u\cos^2\theta}{\sqrt{u^2\cos^2\theta+\sin^2\theta}}\right) \geq 0.$$

It would therefore suffice to show that $f(23/7,\theta)\geq 0$. This is equivalent to proving that

$$(4+4\sqrt{2})(\sin\theta+\cos\theta)+(26-4\sqrt{2}) \geq \sqrt{23^2\sin^2\theta+7^2\cos^2\theta}+\sqrt{23^2\cos^2\theta+7^2\sin^2\theta}.$$

After squaring both sides and simplifying this reduces to showing that

$$(24+16\sqrt{2})(\sin\theta+\cos\theta)^2+(72+88\sqrt{2})(\sin\theta+\cos\theta)+(65-104\sqrt{2}) \geq$$
$$\geq \sqrt{(23^4+7^4)\sin^2\theta\cos^2\theta+23^2\cdot 7^2(\sin^4\theta+\cos^4\theta)}.$$

Denote $x := \sin\theta+\cos\theta$. Note that $1\leq x\leq\sqrt{2}$.

It is then easy to see that $\sin\theta\cos\theta = \frac{x^2-1}{2}$ and therefore $\sin^4\theta+cos^4\theta = 1-\frac{(x^2-1)^2}{2}$.

The inequality to prove becomes

$$(24+16\sqrt{2})x^2+(72+88\sqrt{2})x+(65-104\sqrt{2}) \geq \sqrt{57600x^4-115200x^2+83521}.$$

Note that the left hand term is positive since $x\geq 1$. Squaring both terms once again and moving all terms to the left side we obtain that

$$(\sqrt{2}-x)\cdot(x-1)\cdot\Big(19364x^2+(17084\sqrt{2}+16188)x+(9946\sqrt{2}+4903)\Big)\geq 0,$$

which is certainly true since $1\leq x\leq\sqrt{2}$.

This completes the proof. □

Fact 6.2. *Suppose that* $0=r_1\leq r_2\leq\ldots r_{n-1}\leq r_n\leq 7/8$ *is a nondecreasing sequence of nonnegative numbers and* $\sum_{k=1}^{n} r_k = nR$ *is fixed. Then*

$$\sum_{k=1}^{n} 2kr_k \leq n^2\left(2R-\frac{8}{7}R^2\right)+nR.$$

Proof. Regard $\sum_{k=1}^{n} 2kr_k$ as a continuous function of variables $r_1,r_2,\ldots,r_n$ over the compact set

$\{(r_1,r_2,,\ldots,r_n)|\sum_{k=2}^{n} r_k = nR\}\subset[0,7/8]^n$. It follows that this function reaches its absolute maximum in this set. Assume that there exist indices $i<j$ such that $0<r_i\leq r_j<7/8$. Then for $\epsilon>0$ sufficiently small, replacing r_i by $r_i-\epsilon$ and r_j by $r_j+\epsilon$ will leave $\sum_{k=1}^{n} r_k$ unchanged, while $\sum_{k=1}^{n} 2kr_k$ will increase. Hence, one can assume that all r_i-s (with the possible exception of one) are either 0 or 7/8.

Suppose that $r_1=r_2=\ldots=r_{m-1}=0, r_{m+1}=r_{m+2}=\ldots=7/8$ and $0\leq r_m\leq 7/8$. From $\sum_{k=1}^{n} r_k = nR$ it follows that $r_m+7(n-m)/8=nR$, from which follows

$$n\left(1-\frac{8}{7}R\right)\leq m\leq n\left(1-\frac{8}{7}R\right)+1. \tag{50}$$

It follows that

$$\sum_{k=1}^{n} 2kr_k = 2mr_m + \frac{7}{8}\sum_{k=m+1}^{n} 2k = \frac{7}{8}m^2 + \left(2nR - \frac{7}{4}n - \frac{7}{8}\right)m + \frac{7}{8}(n^2+n)$$

Since this is a quadratic in m with positive leading coefficient, it will reach the maximum value when $m = n(1-8R/7)$ or $m = n(1-8R/7)+1$. Substituting m for any of these values produces the desired result. □

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