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The Euler Line as a Radical Axis of Coaxal Circles

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1. Introduction

We give a proof of the statement made in Euclid #8404.

2. Results

Theorem 2.1. *Let ABC be a triangle and $A'B'C'$ the medial triangle.*
Let N be the nine point center of ABC.
Let A_b be the reflection of A about line $A'C'$.
Define A_c, B_a, B_c, C_a, and C_b cyclically.
Let H_a be the orthocenter of $A'B_cC_b$.
Define H_b and H_c cyclically.
Then circles ANH_a, BNH_b, and CNH_c are coaxal and their radical line is the Euler line of ABC.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
A' has coordinates $((x_C + 1)/2, y_C/2)$.
B' has coordinates $(x_C/2, y_C/2)$.
C' has coordinates $(1/2, 0)$.
N has coordinates $(x_C/2 + 1/4, (-x_C^2 + x_C + y_C^2)/(4 * y_C))$.
A_b has coordinates
$(2 * (y_C^2/(2 * (x_C^2 + y_C^2))), 2 * (-(x_C * y_C)/(2 * (x_C^2 + y_C^2))))$.
A_c has coordinates $(0, y_C)$.

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B_a has coordinates
$(2*((2*x_C^2-4*x_C+y_C^2+2)/(2*(x_C^2-2*x_C+y_C^2+1)))-1, 2*((y_C*(x_C-1))/(2*(x_C^2-2*x_C+y_C^2+1))))$.
B_c has coordinates $(1, y_C)$.
C_a has coordinates
$(2*((2*x_C^3-4*x_C^2+2*x_C*y_C^2+2*x_C-y_C^2)/(2*(x_C^2-2*x_C+y_C^2+1)))-x_C, 2*((y_C*(2*x_C^2-3*x_C+2*y_C^2+1))/(2*(x_C^2-2*x_C+y_C^2+1)))-y_C)$.
C_b has coordinates
$(2*((2*x_C^3+2*x_C*y_C^2+y_C^2)/(2*(x_C^2+y_C^2)))-x_C, 2*((y_C*(2*x_C^2-x_C+2*y_C^2))/(2*(x_C^2+y_C^2)))-y_C)$.
H_a has coordinates $((x_C^3+x_C*y_C^2+2*y_C^2)/(2*(x_C^2+y_C^2)), (-x_C^4+x_C^3+x_C^2*y_C^2-x_C*y_C^2+2*y_C^4)/(2*y_C*(x_C^2+y_C^2)))$.
H_b has coordinates $(-(-x_C^3+x_C^2-x_C*y_C^2+x_C+y_C^2-1)/(2*(x_C^2-2*x_C+y_C^2+1)), (-x_C^4+3*x_C^3+x_C^2*y_C^2-3*x_C^2-x_C*y_C^2+x_C+2*y_C^4)/(2*y_C*(x_C^2-2*x_C+y_C^2+1)))$.
H_c has coordinates
$((x_C^5-2*x_C^4+2*x_C^3*y_C^2+x_C^3-2*x_C^2*y_C^2+x_C*y_C^4-x_C*y_C^2+y_C^2)/((x_C^2+y_C^2)*(x_C^2-2*x_C+y_C^2+1)), ((-x_C^2+x_C+y_C^2)*(x_C^4-2*x_C^3+2*x_C^2*y_C^2+x_C^2-2*x_C*y_C^2+y_C^4-y_C^2))/(2*y_C*(x_C^2+y_C^2)*(x_C^2-2*x_C+y_C^2+1)))$.
Circle ANH_a is given by the equation
$(x-(-(2*y_C^6-(x_C+1)*(3*x_C-2)*y_C^4-(x_C-1)*x_C*(4*x_C^2+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*x_C*y_C^2*(y_C^2+x_C^2-1))))^2+(y-((y_C^2*((5*x_C+2)*y_C^2+2*x_C^3-2*x_C^2+x_C+2)-(x_C-1)^2*x_C^2*(3*x_C+2))/(8*x_C*y_C*(y_C^2+x_C^2-1))))^2-((-(2*y_C^6-(x_C+1)*(3*x_C-2)*y_C^4-(x_C-1)*x_C*(4*x_C^2+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*x_C*y_C^2*(y_C^2+x_C^2-1))))^2-(((y_C^2*((5*x_C+2)*y_C^2+2*x_C^3-2*x_C^2+x_C+2)-(x_C-1)^2*x_C^2*(3*x_C+2))/(8*x_C*y_C*(y_C^2+x_C^2-1))))^2=0$.
Circle BNH_b is given by the equation
$(x-(-(2*y_C^6-(3*x_C^2+x_C-6)*y_C^4-(x_C-1)*x_C*(2*x_C-3)*(2*x_C+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*(x_C-1)*y_C^2*(y_C^2+(x_C-2)*x_C))))^2+(y-((y_C^2*((5*x_C-7)*y_C^2+2*x_C^3-4*x_C^2+3*x_C-3)-(x_C-1)^2*x_C^2*(3*x_C-5))/(8*(x_C-1)*y_C*(y_C^2+(x_C-2)*x_C))))^2-(1-(-(2*y_C^6-(3*x_C^2+x_C-6)*y_C^4-(x_C-1)*x_C*(2*x_C-3)*(2*x_C+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*(x_C-1)*y_C^2*(y_C^2+(x_C-2)*x_C))))^2-(((y_C^2*((5*x_C-7)*y_C^2+2*x_C^3-4*x_C^2+3*x_C-3)-(x_C-1)^2*x_C^2*(3*x_C-5))/(8*(x_C-1)*y_C*(y_C^2+(x_C-2)*x_C))))^2=0$.
Circle CNH_c is given by the equation
$(x-(-(3*y_C^6-(11*x_C^2-3*x_C-7)*y_C^4-(x_C-1)*x_C*(15*x_C^2-7*x_C-3)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)/(8*(2*x_C-1)*y_C^2*(y_C^2+(x_C-1)*x_C))))^2+(y-((y_C^2*(3*y_C^2+2*x_C^2-2*x_C+1)-(x_C-1)^2*x_C^2)/(4*y_C*(y_C^2+(x_C-1)*x_C))))^2-(x_C-(-(3*y_C^6-(11*x_C^2-3*x_C-7)*y_C^4-(x_C-1)*x_C*(15*x_C^2-7*x_C-3)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)/(8*(2*x_C-1)*y_C^2*(y_C^2+(x_C-1)*x_C))))^2-(y_C-((y_C^2*(3*y_C^2+2*x_C^2-2*x_C+1)-(x_C-1)^2*x_C^2)/(4*y_C*(y_C^2+(x_C-1)*x_C))))^2=0$.
The Euler line of ABC is given by the equation
$y=x_C*(1-x_C)/y_C+(x-x_C)*(y_C/3-(x_C*(1-x_C)/y_C))/((x_C+1)/3-x_C)$.
Consider the point $X(46450)$ whose coordinates are given by
$(-(y_C^2*((3*x_C-2)*y_C^2-2*x_C^3-2*x_C^2+7*x_C-2)-(x_C-1)^2*x_C^2*(5*x_C+2))/(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2), (2*y_C^6+(3*x_C-2)*(3*x_C-1)*y_C^4+(x_C-1)*x_C*(4*x_C^2-4*x_C+5)*y_C^2-3*x_C^6+9*x_C^5-9*x_C^4+3*x_C^3)/(y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)))$.
We verify that $X(46450)$ lies on the Euler line since we calculate
$x_C*(1-x_C)/y_C+(-(y_C^2*((3*x_C-2)*y_C^2-2*x_C^3-2*x_C^2+7*x_C-2)-$

$(x_C-1)^2*x_C^2*(5*x_C+2))/(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)-x_C)*(y_C/3-(x_C*(1-x_C)/y_C))/((x_C+1)/3-x_C)-((2*y_C^6+(3*x_C-2)*(3*x_C-1)*y_C^4+(x_C-1)*x_C*(4*x_C^2-4*x_C+5)*y_C^2-3*x_C^6+9*x_C^5-9*x_C^4+3*x_C^3)/(y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)))=0.$
We then verify that $X(46450)$ lies on circles ANH_a, BNH_b, and CNH_c since we calculate
$(-(y_C^2*((3*x_C-2)*y_C^2-2*x_C^3-2*x_C^2+7*x_C-2)-(x_C-1)^2*x_C^2*(5*x_C+2))/(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)-(-(2*y_C^6-(x_C+1)*(3*x_C-2)*y_C^4-(x_C-1)*x_C*(4*x_C^2+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*x_C*y_C^2*(y_C^2+x_C^2-1))))^2+((2*y_C^6+(3*x_C-2)*(3*x_C-1)*y_C^4+(x_C-1)*x_C*(4*x_C^2-4*x_C+5)*y_C^2-3*x_C^6+9*x_C^5-9*x_C^4+3*x_C^3)/(y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((y_C^2*((5*x_C+2)*y_C^2+2*x_C^3-2*x_C^2+x_C+2)-(x_C-1)^2*x_C^2*(3*x_C+2))/(8*x_C*y_C*(y_C^2+x_C^2-1))))^2-((-(2*y_C^6-(x_C+1)*(3*x_C-2)*y_C^4-(x_C-1)*x_C*(4*x_C^2+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*x_C*y_C^2*(y_C^2+x_C^2-1))))^2-(((y_C^2*((5*x_C+2)*y_C^2+2*x_C^3-2*x_C^2+x_C+2)-(x_C-1)^2*x_C^2*(3*x_C+2))/(8*x_C*y_C*(y_C^2+x_C^2-1))))^2=0,$
$(-(y_C^2*((3*x_C-2)*y_C^2-2*x_C^3-2*x_C^2+7*x_C-2)-(x_C-1)^2*x_C^2*(5*x_C+2))/(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)-(-(2*y_C^6-(3*x_C^2+x_C-6)*y_C^4-(x_C-1)*x_C*(2*x_C-3)*(2*x_C+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*(x_C-1)*y_C^2*(y_C^2+(x_C-2)*x_C))))^2+((2*y_C^6+(3*x_C-2)*(3*x_C-1)*y_C^4+(x_C-1)*x_C*(4*x_C^2-4*x_C+5)*y_C^2-3*x_C^6+9*x_C^5-9*x_C^4+3*x_C^3)/(y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((y_C^2*((5*x_C-7)*y_C^2+2*x_C^3-4*x_C^2+3*x_C-3)-(x_C-1)^2*x_C^2*(3*x_C-5))/(8*(x_C-1)*y_C*(y_C^2+(x_C-2)*x_C))))^2-(1-(-(2*y_C^6-(3*x_C^2+x_C-6)*y_C^4-(x_C-1)*x_C*(2*x_C-3)*(2*x_C+3)*y_C^2+x_C^6-3*x_C^5+3*x_C^4-x_C^3)/(8*(x_C-1)*y_C^2*(y_C^2+(x_C-2)*x_C))))^2-(((y_C^2*((5*x_C-7)*y_C^2+2*x_C^3-4*x_C^2+3*x_C-3)-(x_C-1)^2*x_C^2*(3*x_C-5))/(8*(x_C-1)*y_C*(y_C^2+(x_C-2)*x_C))))^2=0,$
and
$(-(y_C^2*((3*x_C-2)*y_C^2-2*x_C^3-2*x_C^2+7*x_C-2)-(x_C-1)^2*x_C^2*(5*x_C+2))/(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2)-(-(3*y_C^6-(11*x_C^2-3*x_C-7)*y_C^4-(x_C-1)*x_C*(15*x_C^2-7*x_C-3)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)/(8*(2*x_C-1)*y_C^2*(y_C^2+(x_C-1)*x_C))))^2+((2*y_C^6+(3*x_C-2)*(3*x_C-1)*y_C^4+(x_C-1)*x_C*(4*x_C^2-4*x_C+5)*y_C^2-3*x_C^6+9*x_C^5-9*x_C^4+3*x_C^3)/(y_C*(y_C^2*(y_C^2+10*x_C^2-10*x_C+1)+9*(x_C-1)^2*x_C^2))-((y_C^2*(3*y_C^2+2*x_C^2-2*x_C+1)-(x_C-1)^2*x_C^2)/(4*y_C*(y_C^2+(x_C-1)*x_C))))^2-(x_C-(-(3*y_C^6-(11*x_C^2-3*x_C-7)*y_C^4-(x_C-1)*x_C*(15*x_C^2-7*x_C-3)*y_C^2-x_C^6+3*x_C^5-3*x_C^4+x_C^3)/(8*(2*x_C-1)*y_C^2*(y_C^2+(x_C-1)*x_C))))^2-(y_C-((y_C^2*(3*y_C^2+2*x_C^2-2*x_C+1)-(x_C-1)^2*x_C^2)/(4*y_C*(y_C^2+(x_C-1)*x_C))))^2=0$
as desired. □

References

[1] Weisstein, Eric W. "Euler Line." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/EulerLine.html

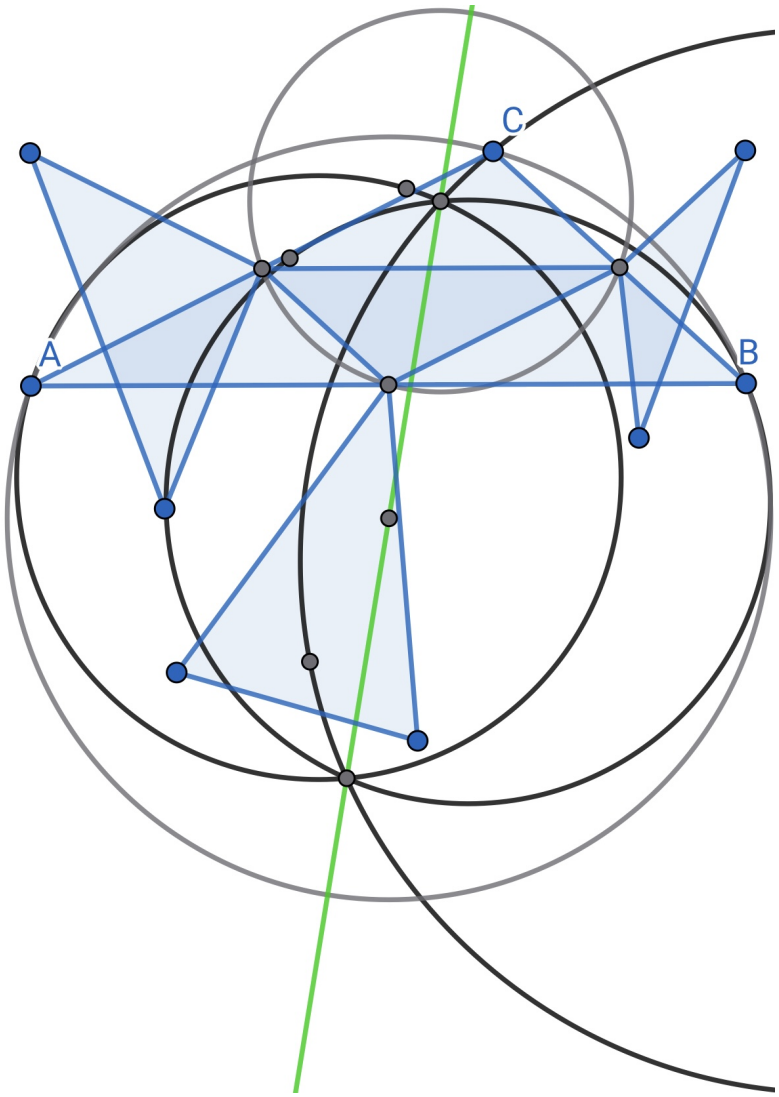

FIGURE 1. Illustration.