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# Another Generalization of the Fuhrmann Circle

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## 1. Introduction

We give a generalization of the Fuhrmann circle which is different than the Hagge circle.

## 2. Results

**Theorem 2.1.** *Let $A, B$, and $C$ be the vertices of a given triangle.*
*Let $P$ be a point in the plane.*
*Let $A'B'C'$ be the circumcevian triangle of $P$.*
*Let $A''B''C''$ be the reflection triangle of $A'B'C'$ about the medial triangle.*
*Then the circumcircle of $A''B''C''$ is identical to the circle whose diameter is the chord connecting the orthocenter and the anticomplement of $P$.*

*Proof.* Without loss of generality, we will work in Cartesian coordinates.
$A$ has coordinates $(0, 0)$.
$B$ has coordinates $(1, 0)$.
$C$ has coordinates $(x_C, y_C)$.
$P$ has coordinates $(x_P, y_P)$.
$A'$ has coordinates
$((x_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)), (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)))$.
$B'$ has coordinates
$((y_P * (y_C * (y_P + (x_P - 1) * y_C) + (x_C - 1) * x_C * (x_P - 1)))/(y_C * (y_P^2 + (x_P - 1)^2)), (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P - (x_P - 1) * y_C))/(y_C * (y_P^2 + (x_P - 1)^2)))$.
$C'$ has coordinates

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$(((x_C * y_P - x_P * y_C) * (y_C * (y_P - y_C) + (x_C - 1) * (x_P - x_C)))/(y_C * ((y_P - y_C)^2 + (x_P - x_C)^2)), (((x_C - 1) * y_P - (x_P - 1) * y_C) * (x_C * y_P - x_P * y_C))/(y_C * ((y_P - y_C)^2 + (x_P - x_C)^2)))$.
$A''$ has coordinates
$(x_C + 1 - (x_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)), y_C - (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)))$.
$B''$ has coordinates
$(x_C - (y_P * (y_C * (y_P + (x_P - 1) * y_C) + (x_C - 1) * x_C * (x_P - 1)))/(y_C * (y_P^2 + (x_P - 1)^2)), y_C - (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P - (x_P - 1) * y_C))/(y_C * (y_P^2 + (x_P - 1)^2)))$.
$C''$ has coordinates
$(1 - ((x_C * y_P - x_P * y_C) * (y_C * (y_P - y_C) + (x_C - 1) * (x_P - x_C)))/(y_C * ((y_P - y_C)^2 + (x_P - x_C)^2)), -(((x_C - 1) * y_P - (x_P - 1) * y_C) * (x_C * y_P - x_P * y_C))/(y_C * ((y_P - y_C)^2 + (x_P - x_C)^2)))$.
The orthocenter has coordinates $(x_C, x_C * (1 - x_C)/y_C)$.
The centroid has coordinates $((x_C + 1)/3, y_C/3)$.
The anticomplement of $P$ has coordinates $(-2 * x_P + x_C + 1, y_C - 2 * y_P)$.
Consider the circle whose diameter is the chord connecting the orthocenter and the anticomplement of $P$, whose equation is given by
$(x-(-2*x_P+2*x_C+1)/2)^2+(y-(x_C*(1-x_C)/y_C+y_C-2*y_P)/2)^2-(x_C-(-2*x_P+2*x_C+1)/2)^2-(x_C*(1-x_C)/y_C-(x_C*(1-x_C)/y_C+y_C-2*y_P)/2)^2 = 0$.
We verify that $A''$, $B''$, and $C''$ lie on this circle since we calculate
$(x_C + 1 - (x_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)) - (-2 * x_P + 2 * x_C + 1)/2)^2 + (y_C - (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)) - (x_C * (1 - x_C)/y_C + y_C - 2 * y_P)/2)^2 - (x_C - (-2 * x_P + 2 * x_C + 1)/2)^2 - (x_C * (1 - x_C)/y_C - (x_C * (1 - x_C)/y_C + y_C - 2 * y_P)/2)^2 = 0$,
$(x_C - (y_P * (y_C * (y_P + (x_P - 1) * y_C) + (x_C - 1) * x_C * (x_P - 1)))/(y_C * (y_P^2 + (x_P - 1)^2)) - (-2*x_P+2*x_C+1)/2)^2 + (y_C - (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P - (x_P - 1) * y_C))/(y_C * (y_P^2 + (x_P - 1)^2)) - (x_C * (1 - x_C)/y_C + y_C - 2 * y_P)/2)^2 - (x_C - (-2 * x_P + 2 * x_C + 1)/2)^2 - (x_C * (1 - x_C)/y_C - (x_C * (1 - x_C)/y_C + y_C - 2 * y_P)/2)^2 = 0$,
and
$(1-((x_C*y_P-x_P*y_C)*(y_C*(y_P-y_C)+(x_C-1)*(x_P-x_C)))/(y_C*((y_P-y_C)^2+(x_P-x_C)^2))-(-2*x_P+2*x_C+1)/2)^2+(-(((x_C-1)*y_P-(x_P-1)*y_C)*(x_C*y_P-x_P*y_C))/(y_C*((y_P-y_C)^2+(x_P-x_C)^2))-(x_C*(1-x_C)/y_C+y_C-2*y_P)/2)^2-(x_C-(-2*x_P+2*x_C+1)/2)^2-(x_C*(1-x_C)/y_C-(x_C*(1-x_C)/y_C+y_C-2*y_P)/2)^2 = 0$
as desired. □

The Fuhrmann circle follows as the special case of this where we take $P$ to be the incenter.
The result in [1] is a special case of this where we take $P$ to be the centroid.

## REFERENCES

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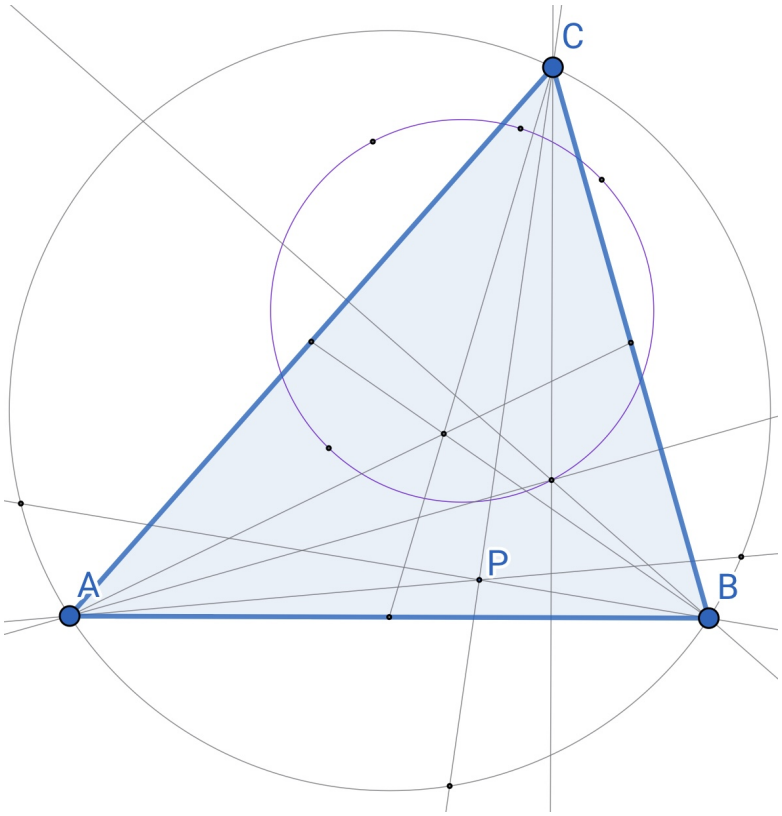


Figure 1. Illustration.