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A Line and Circle in the Context of a Parabola

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1. Introduction

We present a line and circle that both pass through the focus of a given parabola.

2. Results

Theorem 2.1. *Let w be a parabola with focus A.*
Let B and C be points on w.
Let the tangents to B and C intersect at D.
Let the tangent to B intersect the normal to C at E.
Let the tangent to C intersect the normal to B and F.
Let G be the midpoint of E and F.
Then,
1. A, D, and G are collinear and
2. A, B, C, and G are concyclic.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates (x_A, y_A).
The directrix of w can be set to $y = 0$.
w is given by the equation $y = ((x - x_A)^2 + y_A^2)/(2 * y_A)$.
w has derivative $\frac{dy}{dx} = (x - x_A)/y_A$.
B has coordinates
$(x_B, ((x_B - x_A)^2 + y_A^2)/(2 * y_A))$.
C has coordinates
$(x_C, ((x_C - x_A)^2 + y_A^2)/(2 * y_A))$.

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The tangent to B is given by the equation
$y = ((x_B - x_A) * (2 * x - x_B - x_A) + y_A^2)/(2 * y_A)$.
The tangent to C is given by the equation
$y = ((x_C - x_A) * (2 * x - x_C - x_A) + y_A^2)/(2 * y_A)$.
The normal to B is given by the equation
$y = (y_A^2 + (x_B - x_A)^2)/(2 * y_A) - (y_A * (x - x_B))/(x_B - x_A)$.
The normal to C is given by the equation
$y = (y_A^2 + (x_C - x_A)^2)/(2 * y_A) - (y_A * (x - x_C))/(x_C - x_A)$.
D has coordinates
$((x_B + x_C)/2, (y_A^2 + (x_B - x_A) * (x_C - x_A))/(2 * y_A))$.
E has coordinates
$((2*x_C*y_A^2+x_C^3-3*x_A*x_C^2+x_B^2*x_C+2*x_A^2*x_C-x_A*x_B^2)/(2*(y_A^2+(x_B-x_A)*(x_C-x_A))), (y_A^4+(x_B-x_A)*(3*x_C-x_B-2*x_A)*y_A^2+(x_B-x_A)*x_C^3-3*x_A*(x_B-x_A)*x_C^2+3*x_A^2*(x_B-x_A)*x_C-x_A^3*x_B+x_A^4)/(2*y_A*(y_A^2+(x_B-x_A)*(x_C-x_A))))$.
F has coordinates
$((2 * x_B * y_A^2 + x_B * x_C^2 - x_A * x_C^2 + x_B^3 - 3 * x_A * x_B^2 + 2 * x_A^2 * x_B)/(2 * (y_A^2 + (x_B - x_A) * (x_C - x_A))), (y_A^4 - (x_C - x_A) * (x_C - 3 * x_B + 2 * x_A) * y_A^2 + (x_B - x_A)^3 * x_C - x_A * x_B^3 + 3 * x_A^2 * x_B^2 - 3 * x_A^3 * x_B + x_A^4)/(2 * y_A * (y_A^2 + (x_B - x_A) * (x_C - x_A))))$.
G has coordinates
$((2 * (x_C + x_B) * y_A^2 + x_C^3 + (x_B - 4 * x_A) * x_C^2 + (x_B^2 + 2 * x_A^2) * x_C + x_B^3 - 4 * x_A * x_B^2 + 2 * x_A^2 * x_B)/(4 * (y_A^2 + (x_B - x_A) * (x_C - x_A))), (2 * y_A^4 + (x_B - x_A) * (3 * x_C - x_B - 2 * x_A) * y_A^2 - (x_C - x_A) * (x_C - 3 * x_B + 2 * x_A) * y_A^2 + (x_B - x_A) * x_C^3 - 3 * x_A * (x_B - x_A) * x_C^2 + (x_B - x_A)^3 * x_C + 3 * x_A^2 * (x_B - x_A) * x_C - x_A * x_B^3 + 3 * x_A^2 * x_B^2 - 4 * x_A^3 * x_B + 2 * x_A^4)/(4 * y_A * (y_A^2 + (x_B - x_A) * (x_C - x_A))))$.
Consider the line whose equation is given by
$y = y_A + (x - x_A) * ((y_A^2 + (x_B - x_A) * (x_C - x_A))/(2 * y_A) - y_A)/((x_B + x_C)/2 - x_A)$.
We verify that A, D, and G lie on this line since we calculate
$y_A + (x_A - x_A) * ((y_A^2 + (x_B - x_A) * (x_C - x_A))/(2 * y_A) - y_A)/((x_B + x_C)/2 - x_A) - y_A = 0$,
$y_A + ((x_B + x_C)/2 - x_A) * ((y_A^2 + (x_B - x_A) * (x_C - x_A))/(2 * y_A) - y_A)/((x_B + x_C)/2 - x_A) - ((y_A^2 + (x_B - x_A) * (x_C - x_A))/(2 * y_A)) = 0$,
$y_A + ((2 * (x_C + x_B) * y_A^2 + x_C^3 + (x_B - 4 * x_A) * x_C^2 + (x_B^2 + 2 * x_A^2) * x_C + x_B^3 - 4 * x_A * x_B^2 + 2 * x_A^2 * x_B)/(4 * (y_A^2 + (x_B - x_A) * (x_C - x_A))) - x_A) * ((y_A^2 + (x_B - x_A) * (x_C - x_A))/(2 * y_A) - y_A)/((x_B + x_C)/2 - x_A) - ((2 * y_A^4 + (x_B - x_A) * (3 * x_C - x_B - 2 * x_A) * y_A^2 - (x_C - x_A) * (x_C - 3 * x_B + 2 * x_A) * y_A^2 + (x_B - x_A) * x_C^3 - 3 * x_A * (x_B - x_A) * x_C^2 + (x_B - x_A)^3 * x_C + 3 * x_A^2 * (x_B - x_A) * x_C - x_A * x_B^3 + 3 * x_A^2 * x_B^2 - 4 * x_A^3 * x_B + 2 * x_A^4)/(4 * y_A * (y_A^2 + (x_B - x_A) * (x_C - x_A)))) = 0$ as desired.
Now, consider the circle whose equation is given by
$(x-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2+(y-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2-(x_A-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2-$

$(y_A-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2=0.$
We verify that A, B, C, and G lie on this circle since we calculate
$(x_A-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2+(y_A-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2-(x_A-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2-(y_A-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2=0.$
$(x_B-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2+(((x_B-x_A)^2+y_A^2)/(2*y_A)-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2-(x_A-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2-(y_A-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2=0,$
$(x_C-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2+(((x_C-x_A)^2+y_A^2)/(2*y_A)-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2-(x_A-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2-(y_A-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2=0,$
and
$((2*(x_C+x_B)*y_A^2+x_C^3+(x_B-4*x_A)*x_C^2+(x_B^2+2*x_A^2)*x_C+x_B^3-4*x_A*x_B^2+2*x_A^2*x_B)/(4*(y_A^2+(x_B-x_A)*(x_C-x_A)))-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+$

$4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2)))^2+((2*y_A^4+(x_B-x_A)*(3*x_C-x_B-2*x_A)*y_A^2-(x_C-x_A)*(x_C-3*x_B+2*x_A)*y_A^2+(x_B-x_A)*x_C^3-3*x_A*(x_B-x_A)*x_C^2+(x_B-x_A)^3*x_C+3*x_A^2*(x_B-x_A)*x_C-x_A*x_B^3+3*x_A^2*x_B^2-4*x_A^3*x_B+2*x_A^4)/(4*y_A*(y_A^2+(x_B-x_A)*(x_C-x_A)))-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2-(x_A-(-(((x_B-x_A)^2-y_A^2)*(y_A^2+(x_C-x_A)^2)^2+2*y_A^2*(y_A^2+(x_B-x_A)^2)^2+(y_A^2+(x_C-x_A)^2)*(4*y_A^2*(y_A^2-x_B^2+x_A^2)-(y_A^2+(x_B-x_A)^2)^2)+4*y_A^2*((-y_A^2+x_C^2-x_A^2)*(y_A^2+(x_B-x_A)^2)+2*(x_B^2-x_C^2)*y_A^2))/(8*y_A^2*((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)+2*(x_C-x_B)*y_A^2))))^2-(y_A-((((x_B-x_A)*(y_A^2+(x_C-x_A)^2)^2+(x_A-x_C)*(y_A^2+(x_B-x_A)^2)^2)/(4*y_A^2)+(x_C-x_B)*y_A^2+(x_B-x_A)*x_C^2+(x_A^2-x_B^2)*x_C+x_A*x_B^2-x_A^2*x_B)/(2*(((x_B-x_A)*(y_A^2+(x_C-x_A)^2)+(x_A-x_C)*(y_A^2+(x_B-x_A)^2))/(2*y_A)+(x_C-x_B)*y_A))))^2=0$
as desired. □

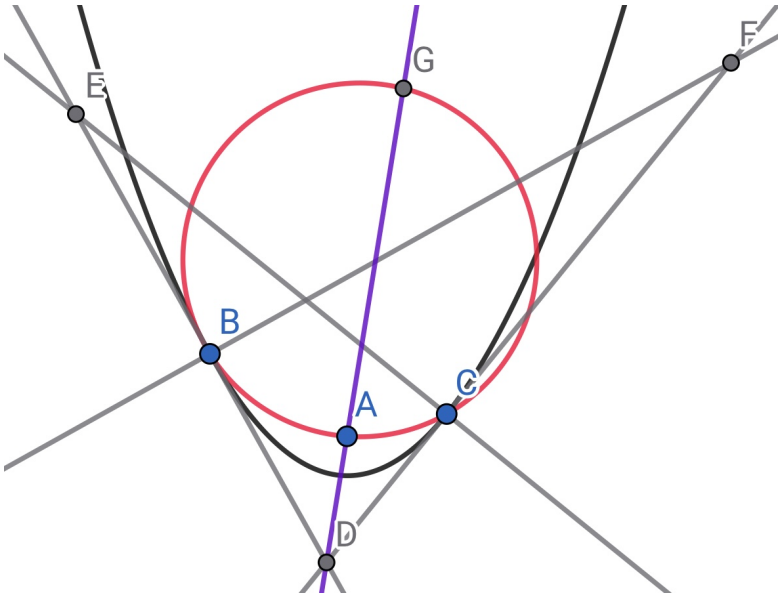

Figure 1. Illustration.

References

[1] Warren, B.L., "Tangents and a Parabola", *International Journal of Computer Discovered Mathematics*, under review.