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Generalization of the Incenter, Gergonne Point, and X(104)

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1. Results

Theorem 1.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be the point on line AC, with $\angle CAB = \angle DAB$, such that $AD = AB$.
Let E be the point on line AB, with $\angle CAB = \angle CAE$, such that $AE = AC$.
Let F be the point on line BC, with $\angle CBA = \angle FBA$, such that $BF = AB$.
Let G be the point on line AB, with $\angle CBA = \angle CBG$, such that $BG = BC$.
Let H be the point on line BC, with $\angle ACB = \angle ACH$, such that $CH = AC$.
Let I be the point on line AC, with $\angle ACB = \angle ICB$, such that $CI = BC$.
Let J be the midpoint of E and G.
Let K be the midpoint of D and I.
Let L be the midpoint of F and H.
Let M be the circumcenter of A, G, and I.
Let N be the circumcenter of C, D, and F.
Let O be the circumcenter of B, E, and H.
Let P be a point on line AM.
Let Q be the point on line BO such that $BQ : OQ = AP : MP$.
Let R be the point on line CN such that $CR : NR = AP : MP$.
Let S be a point on line AL.
Let T be the point on line BK such that $BT : KT = AS : LS$.
Let U be the point on line CJ such that $CU : JU = AS : LS$.
Then lines PS, QT, and RU are concurrent.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates $(\frac{x_C}{\sqrt{x_C^2+y_C^2}}, \frac{y_C}{\sqrt{x_C^2+y_C^2}})$.
E has coordinates $(\sqrt{x_C^2+y_C^2}, 0)$.
F has coordinates $(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1, \frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})$.
G has coordinates $(1-\sqrt{(x_C-1)^2+y_C^2}, 0)$.
H has coordinates $(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}, y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))$.
I has coordinates $(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}), y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))$.
J has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2}+\sqrt{x_C^2+y_C^2})/2, 0)$.
K has coordinates $(\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}, \frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$.
L has coordinates $((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2, (\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)$.
M has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
N has coordinates $(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}, -\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})$.
O has coordinates $((1+\sqrt{x_C^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
P has coordinates
$(x_P, x_P*(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})/((1-\sqrt{(x_C-1)^2+y_C^2})/2))$.
Q has coordinates
$(1+x_P*(((1+\sqrt{x_C^2+y_C^2})/2)-1)/((1-\sqrt{(x_C-1)^2+y_C^2})/2),$
$x_P*(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})/((1-\sqrt{(x_C-1)^2+y_C^2})/2))$.
R has coordinates
$(x_C+x_P*((\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2})-x_C)/((1-\sqrt{(x_C-1)^2+y_C^2})/2),$
$(x_P*((-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})-y_C))/((1-\sqrt{(x_C-1)^2+y_C^2})/2)+$
$y_C)$.
S has coordinates $(x_S, x_S*((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)/((1+x_C+$
$\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2))$.
T has coordinates
$(1+x_S*((\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})-1)/((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2), x_S*$
$(\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})/((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2))$.
U has coordinates
$(x_C+x_S*(((1-\sqrt{(x_C-1)^2+y_C^2}+\sqrt{x_C^2+y_C^2})/2)-x_C)/((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2),$
$(x_S*((0)-y_C))/((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2)+y_C)$.
Line PS is given by the equation
$y=-(((x_S*(y_C/((x_C-1)^2+y_C^2)^{1/2}-y_C*((x_C^2+y_C^2)^{1/2}/((x_C-1)^2+y_C^2)^{1/2}-$
$1)))/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1)+(x_P*$

$(((x_C-1)^2+y_C^2)^{1/2}*(x_C-(x_C^2+y_C^2)^{1/2})-x_C+x_C^2+y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1)))*(x-x_P))/(x_P-x_S)-(x_P*(((x_C-1)^2+y_C^2)^{1/2}*(x_C-(x_C^2+y_C^2)^{1/2})-x_C+x_C^2+y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1))$.
Line QT is given by the equation
$y=(x_P*(x_C+(x_C^2+y_C^2)^{1/2}*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-x_C^2-y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1))-(((x_P*(x_C+(x_C^2+y_C^2)^{1/2}*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-x_C^2-y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1))-(x_S*y_C*((x_C^2+y_C^2)^{1/2}-((x_C-1)^2+y_C^2)^{1/2}+1))/((x_C^2+y_C^2)^{1/2}*(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1)))*(x+(2*x_P*((x_C^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1)-1))/((2*x_P*((x_C^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1)+(2*x_S*((x_C*((x_C^2+y_C^2)^{1/2}-((x_C-1)^2+y_C^2)^{1/2}+1))/(2*(x_C^2+y_C^2)^{1/2})-1))/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1))$.
Line RU is given by the equation
$y=y_C+(2*x_P*(y_C-(x_C*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-(x_C^2+y_C^2)^{1/2}*(x_C-1)+y_C^2)/(2*y_C)))/(((x_C-1)^2+y_C^2)^{1/2}-1)-(((2*x_S*y_C)/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1)+(2*x_P*(y_C-(x_C*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-(x_C^2+y_C^2)^{1/2}*(x_C-1)+y_C^2)/(2*y_C)))/(((x_C-1)^2+y_C^2)^{1/2}-1))*(x_C-x+(2*x_P*(x_C-(x_C^2+y_C^2)^{1/2}/2+((x_C-1)^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1)))/((2*x_P*(x_C-(x_C^2+y_C^2)^{1/2}/2+((x_C-1)^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1)+(2*x_S*(x_C-(x_C^2+y_C^2)^{1/2}/2+((x_C-1)^2+y_C^2)^{1/2}/2-1/2))/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1))$.
Let $A=-(((x_S*(y_C/((x_C-1)^2+y_C^2)^{1/2}-y_C*((x_C^2+y_C^2)^{1/2}/((x_C-1)^2+y_C^2)^{1/2}-1)))/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1)+(x_P*(((x_C-1)^2+y_C^2)^{1/2}*(x_C-(x_C^2+y_C^2)^{1/2})-x_C+x_C^2+y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1))))/(x_P-x_S)$.
Let $B=-1$ Let $D=-A*x_P-(x_P*(((x_C-1)^2+y_C^2)^{1/2}*(x_C-(x_C^2+y_C^2)^{1/2})-x_C+x_C^2+y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1))$.
Let $E=-(((x_P*(x_C+(x_C^2+y_C^2)^{1/2}*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-x_C^2-y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1))-(x_S*y_C*((x_C^2+y_C^2)^{1/2}-((x_C-1)^2+y_C^2)^{1/2}+1))/((x_C^2+y_C^2)^{1/2}*(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1))))/(((2*x_P*((x_C^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1)+(2*x_S*((x_C*((x_C^2+y_C^2)^{1/2}-((x_C-1)^2+y_C^2)^{1/2}+1))/(2*(x_C^2+y_C^2)^{1/2})-1))/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1)))$.
Let $F=-1$.
Let $G=E*((2*x_P*((x_C^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1)-1)+(x_P*(x_C+(x_C^2+y_C^2)^{1/2}*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-x_C^2-y_C^2))/(y_C*(((x_C-1)^2+y_C^2)^{1/2}-1))$.
Let $H=(((2*x_S*y_C)/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1)+(2*x_P*(y_C-(x_C*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-(x_C^2+y_C^2)^{1/2}*(x_C-1)+y_C^2)/(2*y_C)))/(((x_C-1)^2+y_C^2)^{1/2}-1)))/(((2*x_P*(x_C-(x_C^2+y_C^2)^{1/2}/2+((x_C-1)^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1)+(2*x_S*(x_C-(x_C^2+y_C^2)^{1/2}/2+((x_C-1)^2+y_C^2)^{1/2}/2-1/2))/(x_C-((x_C^2+y_C^2)^{1/2}*(x_C-1)-x_C+1)/((x_C-1)^2+y_C^2)^{1/2}+1)))$.
Let $I=-1$.
Let $J=-H*(x_C+(2*x_P*(x_C-(x_C^2+y_C^2)^{1/2}/2+((x_C-1)^2+y_C^2)^{1/2}/2-1/2))/(((x_C-1)^2+y_C^2)^{1/2}-1))+y_C+(2*x_P*(y_C-(x_C*(x_C+((x_C-1)^2+y_C^2)^{1/2}-1)-(x_C^2+y_C^2)^{1/2}*(x_C-1)+y_C^2)/(2*y_C)))/(((x_C-1)^2+y_C^2)^{1/2}-1)$.
We verify that lines PS, QT, and RU are concurrent since we substitute the above values into the following matrix and calculate that

$$det \begin{bmatrix} A & B & D \\ E & F & G \\ H & I & J \end{bmatrix} = 0$$

as desired.

□

This point of concurrency becomes the incenter when we take $P = M$ and $S = L$, the Gergonne point when we take $P = A$ and $S = L$, and $X(104)$ when we take $P = M$ and $S = A$.

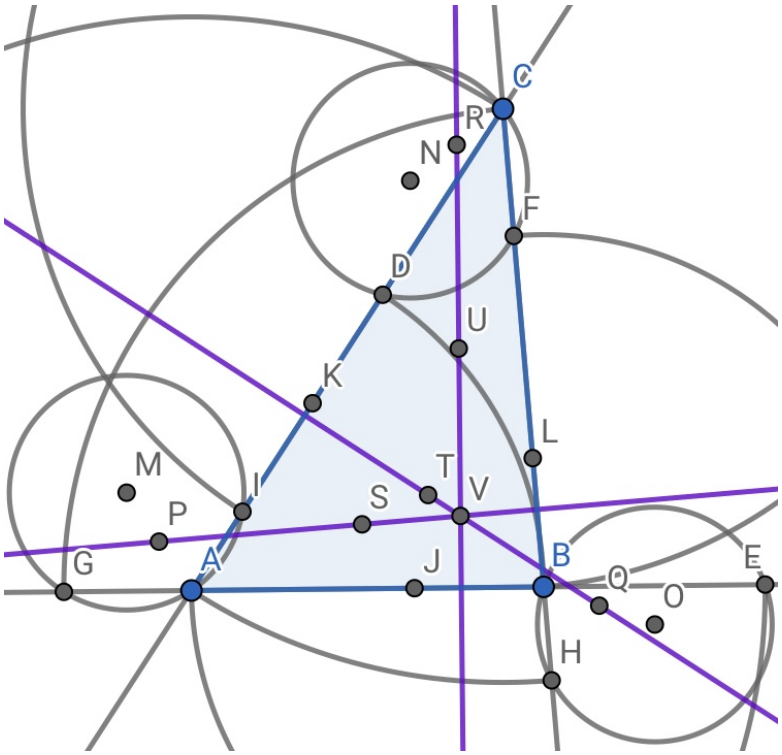

FIGURE 1. Illustration.

REFERENCES

[1] Warren, B.L., "Three Congruent Circles, Six Parallel Lines, X(104), and a New Construction of the Intouch Triangle", *International Journal of Computer Discovered Mathematics*, under review.

[2] Warren, B.L., "The Incenter as a Homothetic Center of the Intouch Triangle", *International Journal of Computer Discovered Mathematics*, under review.