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Coaxal Circles in the Context of the Orthic and Circumcevian Triangles

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1. Introduction

We demonstrate the existence of a set of coaxal circles whose radical line passes through the orthocenter.

2. Results

Theorem 2.1. *Let ABC be a triangle and P a point in the plane.*
Let $A'B'C'$ be the circumcevian triangle of P.
Let $A''B''C''$ be a triangle which is homothetic with the orthic triangle about the orthocenter.
Then circles $AA'A''$, $BB'B''$, and $CC'C''$ are coaxal and both P and the orthocenter lie on their radical axis.

Proof. Without loss of generality, we will work in Cartesian coordinates
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
P has coordinates (x_P, y_P).
The orthocenter has coordinates $(x_C, ((1 - x_C) * x_C)/y_C)$.
A' has coordinates
$((x_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)), (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P + x_P * y_C))/(y_C * (y_P^2 + x_P^2)))$.
B' has coordinates
$((y_P * (y_C * (y_P + (x_P - 1) * y_C) + (x_C - 1) * x_C * (x_P - 1)))/(y_C * (y_P^2 + (x_P -$

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$1)^2)), (y_P * ((y_C^2 + (x_C - 1) * x_C) * y_P - (x_P - 1) * y_C))/(y_C * (y_P^2 + (x_P - 1)^2)))$.
C' has coordinates
$(((x_C * y_P - x_P * y_C) * (y_C * (y_P - y_C) + (x_C - 1) * (x_P - x_C)))/(y_C * ((y_P - y_C)^2 + (x_P - x_C)^2)), (((x_C - 1) * y_P - (x_P - 1) * y_C) * (x_C * y_P - x_P * y_C))/(y_C * ((y_P - y_C)^2 + (x_P - x_C)^2)))$.
A'' has coordinates
$(-(y_C * (y_C^2 * y_H + x_C^2 * y_H - x_C * y_H - x_C * y_C))/(x_C * (y_C^2 + (x_C - 1)^2)), ((x_C - 1) * (y_C^2 * y_H + x_C^2 * y_H - x_C * y_H - x_C * y_C))/(x_C * (y_C^2 + (x_C - 1)^2)))$.
B'' has coordinates
$(-(y_C * (y_C^2 + (x_C - 1) * x_C) * y_H - x_C^3 + x_C^2)/((x_C - 1) * (y_C^2 + x_C^2)), (x_C * ((y_C^2 + (x_C - 1) * x_C) * y_H + (x_C - 1) * y_C))/((x_C - 1) * (y_C^2 + x_C^2)))$.
C'' has coordinates (x_C, y_H).
Circle $AA'A''$ is given by the equation
$(x-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-x_C^2*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*x_P*y_C)/(2*x_C*y_C*(y_C*y_P+(x_C-1)*x_P))))^2+(y-((y_C^2+(x_C-1)*x_C)*(x_C*y_P+x_P*y_H))/(2*x_C*(y_C*y_P+(x_C-1)*x_P)))^2-(-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-x_C^2*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*x_P*y_C)/(2*x_C*y_C*(y_C*y_P+(x_C-1)*x_P))))^2-(-((y_C^2+(x_C-1)*x_C)*(x_C*y_P+x_P*y_H))/(2*x_C*(y_C*y_P+(x_C-1)*x_P)))^2 = 0$.
Circle $BB'B''$ is given by the equation
$(x-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-(x_C-1)*(x_C+1)*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*(x_P-1)*y_C)/(2*(x_C-1)*y_C*(y_C*y_P+x_C*(x_P-1)))))^2+(y-(((y_C^2+(x_C-1)*x_C)*((x_C-1)*y_P+(x_P-1)*y_H))/(2*(x_C-1)*(y_C*y_P+x_C*(x_P-1)))))^2-(1-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-(x_C-1)*(x_C+1)*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*(x_P-1)*y_C)/(2*(x_C-1)*y_C*(y_C*y_P+x_C*(x_P-1)))))^2-(-(((y_C^2+(x_C-1)*x_C)*((x_C-1)*y_P+(x_P-1)*y_H))/(2*(x_C-1)*(y_C*y_P+x_C*(x_P-1)))))^2 = 0$.
Circle $CC'C''$ is given by the equation
$(x - (-((y_C * y_H - x_C^2 + x_C) * y_P - y_C^2 * y_H + (x_C^2 - x_P) * y_C)/(2 * (x_P - x_C) * y_C)))^2 + (y - (y_C + y_H)/2)^2 - (x_C - (-((y_C * y_H - x_C^2 + x_C) * y_P - y_C^2 * y_H + (x_C^2 - x_P) * y_C)/(2 * (x_P - x_C) * y_C)))^2 - (y_C - (y_C + y_H)/2)^2 = 0$.
Consider the line whose equation is given by
$(y_C * y_P + x_C^2 - x_C) * x - x_C * y_C * y_P + (x_C - x_P) * y * y_C + (x_C - x_C^2) * x_P = 0$.
We verify that this line is the radical line of circles $AA'A''$, $BB'B''$, and $CC'C''$ since we calculate
$((((x_C-1)*x_C*y_C*(y_C*y_P+x_C*(x_P-1))*(y_C*y_P+(x_C-1)*x_P)))/(-((y_C^2+(x_C-1)*x_C)*(y_C*y_H-(x_C-1)*x_C)*y_P)))*((x-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-x_C^2*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*x_P*y_C)/(2*x_C*y_C*(y_C*y_P+(x_C-1)*x_P))))^2+(y-((y_C^2+(x_C-1)*x_C)*(x_C*y_P+x_P*y_H))/(2*x_C*(y_C*y_P+(x_C-1)*x_P)))^2-(-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-x_C^2*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*x_P*y_C)/(2*x_C*y_C*(y_C*y_P+(x_C-1)*x_P))))^2-(-((y_C^2+(x_C-1)*x_C)*(x_C*y_P+x_P*y_H))/(2*x_C*(y_C*y_P+(x_C-1)*x_P)))^2-((x-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-(x_C-1)*(x_C+1)*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*(x_P-1)*y_C)/(2*(x_C-1)*y_C*(y_C*y_P+x_C*(x_P-1)))))^2+(y-(((y_C^2+(x_C-1)*x_C)*((x_C-1)*y_P+(x_P-1)*y_H))/(2*(x_C-1)*(y_C*y_P+x_C*(x_P-1)))))^2-(1-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-(x_C-1)*(x_C+1)*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*(x_P-1)*y_C)/(2*(x_C-1)*y_C*(y_C*y_P+x_C*(x_P-1)))))^2-(-(((y_C^2+(x_C-1)*x_C)*((x_C-1)*y_P+(x_P-1)*y_H))/(2*(x_C-1)*(y_C*y_P+x_C*(x_P-1)))))^2))-((y_C*y_P+x_C^2-x_C)*x-x_C*y_C*y_P+(x_C-x_P)*y*y_C+(x_C-x_C^2)*x_P) = 0$
and

$(-((x_C*(x_P-x_C)*y_C*(y_C*y_P+(x_C-1)*x_P)))/((y_C*y_H-x_C^2+x_C)*(x_C*y_P-x_P*y_C)))*((x-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-x_C^2*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*x_P*y_C)/(2*x_C*y_C*(y_C*y_P+(x_C-1)*x_P))))^2+(y-((y_C^2+(x_C-1)*x_C)*(x_C*y_P+x_P*y_H))/(2*x_C*(y_C*y_P+(x_C-1)*x_P)))^2-(-(-((y_C*(y_C^2+(x_C-1)*x_C)*y_H-x_C^2*y_C^2-x_C^4+2*x_C^3-x_C^2)*y_P-(x_C-1)*x_C*x_P*y_C)/(2*x_C*y_C*(y_C*y_P+(x_C-1)*x_P))))^2-(-((y_C^2+(x_C-1)*x_C)*(x_C*y_P+x_P*y_H))/(2*x_C*(y_C*y_P+(x_C-1)*x_P)))^2-((x-(-((y_C*y_H-x_C^2+x_C)*y_P-y_C^2*y_H+(x_C^2-x_P)*y_C)/(2*(x_P-x_C)*y_C)))^2+(y-(y_C+y_H)/2)^2-(x_C-(-((y_C*y_H-x_C^2+x_C)*y_P-y_C^2*y_H+(x_C^2-x_P)*y_C)/(2*(x_P-x_C)*y_C)))^2-(y_C-(y_C+y_H)/2)^2))-((y_C*y_P+x_C^2-x_C)*x-x_C*y_C*y_P+(x_C-x_P)*y*y_C+(x_C-x_C^2)*x_P)=0.$
We then verify that P and the orthocenter lie on this line since we calculate
$(y_C*y_P+x_C^2-x_C)*x_P-x_C*y_C*y_P+(x_C-x_P)*y_P*y_C+(x_C-x_C^2)*x_P=0$
and
$(y_C*y_P+x_C^2-x_C)*x_C-x_C*y_C*y_P+(x_C-x_P)*(((1-x_C)*x_C)/y_C)*y_C+(x_C-x_C^2)*x_P=0$ as desired. □

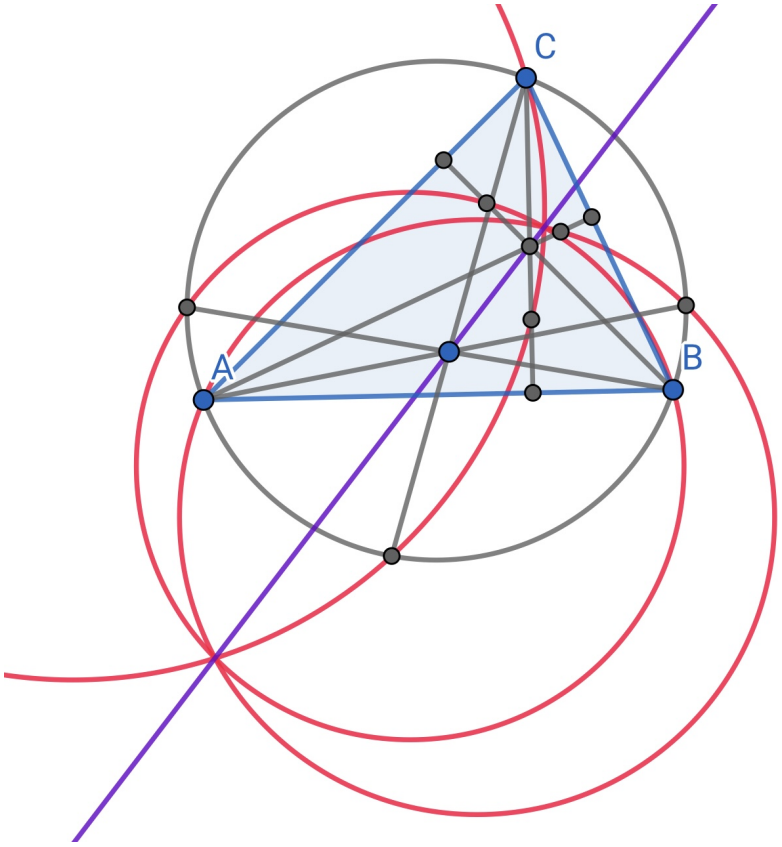

Figure 1. Illustration.

References

[1] Weisstein, Eric W. "Coaxal Circles." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/CoaxalCircles.html