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A new family of numbers to generate potentially infinite number of approximations of π, e and their combinations

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Abstract. Over the years, mathematicians have attempted multiple ways to find simple formulas to approximate π and e to certain degree of accuracy. However, no single pattern or formula is documented as of today to generate various (possibly infinite number of) approximate combinations of π and e. This work introduces one single formula involving a triplet of positive integers (a, b, c) to generate several thousands of approximations of not only π and e, but as many as 16 different combinations of π and e, and small integer multiples of those combinations.

Keywords. anticevian corner product, triangle geometry, remarkable point, computer-discovered mathematics, Euclidean geometry, Discoverer.

Mathematics Subject Classification (2010). 41-04, 65-05.

1. Introduction

π and e are two of the most well-known irrational numbers in Mathematics. Supercomputers have been deployed since several decades and new records of computing more and more decimal digits of π are being set almost every day [1]. There are several yet to be proven results regarding rationality or irrationality of the combinations such as $(\pi + e)$, πe, $\frac{\pi}{e}$ etc.
Several mathematicians have come up with simple elegant approximations for π. One of the most notable approximations is credited to Ramanujan [2] as $\pi \approx \frac{9801}{2206\sqrt{2}}$ which is correct to 6 places of decimals. In 2007, Lucas [3] presented a family of definite integrals which define lower and upper bounds for π in terms of simple

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rational fractions. As recently as in the year 2023, Szyszkowicz [4] has published a paper on improving Archimedes' approximation for π.
Some simple rational approximations of π and e have been proposed by Philip [5]. Pellis [6] has derived hundreds of different approximations connecting π, golden ratio ϕ and e. Irkhin [7] has published interesting explanations of some well-known coincidences such as $e + 2\pi \approx 9$. The work by Janjic [8] has established a relation between π and infinite nested radicals. Further,
Janorkar [9] and Zhu [10] have proposed detailed geometrical constructions to approximate line segments of length π with good accuracy.
Undoubtedly, over the years, mathematicians have attempted multiple ways to find simple formulas to approximate π and e to certain degree of accuracy. However, no single pattern or formula is known as of now to generate various (possibly infinite number of) approximate combinations of π and e. The present work introduces one single formula involving a triplet of positive integers (a, b, c) to generate several thousands of approximations of not only π and e, but as many as 16 different combinations of π and e, and small integer multiples of those combinations. Another interesting aspect of the formula is that it consistently generates approximations which are guaranteed to be correct upto at least 8 places of decimals.

2. Our Work

People interested in numbers often play around with calculators and that has also led to some interesting discoveries. The foundation for this work started when the authors discovered the following approximation while casually fiddling around on a calculator.

$$\frac{73 + \sqrt{8191}}{\sqrt{3618}} = \mathbf{2.71828182}08... \approx e$$

Above approximation matches the Euler's number e upto 9 places of decimals. This inspired the authors to look for more such similar patterns with the help of a simple computer program written in Java. After trying out a bunch of similar patterns, the authors ended up finding a formula which is guaranteed to generate several thousands of approximations of combinations of π and e with an accuracy of at least 8 places of decimals.
Some relevant terms are defined below before presenting the formula.

Definition 2.1 (Approximate integer)**.** A real number x is defined as an approximate integer if $|x - \lfloor x \rceil| \leq 10^{-8}$. Here, $\lfloor x \rceil$ represents the rounded value of x.

Definition 2.2 (Small integer multiple)**.** A small integer multiple of a real number x is defined as px, where p is an integer and $1 \leq p \leq 10$.

Definition 2.3 (Triplet)**.** A group of 3 real numbers (a, b, c) is called a triplet. It obviously represents a point in three-dimensional coordinate system.

Definition 2.4 (Triplet Gap)**.** It is defined as the absolute difference between the a-values of two triplets. The triplet-gap between two triplets (a, b, c) and (m, n, p) is $|a - m|$.

Finally, the formula for generating several thousands of approximations of combinations of π and e using triplets of integers (a, b, c) is as follows:

$$V = \sqrt{\frac{a + \sqrt{b}}{k\sqrt{c}}} \tag{1}$$

Here, a, b and c are the positive integers with the constraints $1 \leq b \leq 10^4$, $1 \leq c \leq 10^4$, and a being unbounded. As a can grow in an unbounded fashion, so to balance the same in the above fraction, the value k is defined as :

$$k = \begin{cases} 1, & \text{if } 1 \leq a \leq 10^4 \\ \frac{1}{2} + \lfloor \frac{a}{10^4} \rfloor, & \text{otherwise} \end{cases}$$

Here, $\lfloor \rfloor$ is the **floor** operator, and $\lfloor x \rfloor$ means the greatest integer which is less than or equal to x.

Equation (1) results in diverse good approximations of simple mathematical expressions involving π and e, and their mathematical combinations. There are surprisingly high number of good approximations with error margin of $\epsilon \leq 10^{-8}$ for the mathematical quantities π, $\sqrt{\pi}$, e, $\sqrt{e}$, $(\pi + e)$, $\sqrt{(\pi + e)}$, π^2, e^2, $\frac{\pi}{e}$, $\sqrt{\frac{\pi}{e}}$, πe, $\sqrt{\pi e}$ and their small integer multiples as well. A small number of approximations are found for $(\pi - e)$, $\sqrt{(\pi - e)}$, $\frac{e}{\pi}$, $\sqrt{\frac{e}{\pi}}$ and their small integer multiples as well.
A total of **158299** approximations have been discovered by exploring the range defined by the constraints $(1 \leq a \leq 10^6, 1 \leq b \leq 10^4, 1 \leq c \leq 10^4)$. Some of these approximations are listed below. The digits marked in bold indicate the matching digits.

Table 1: Some approximations generated by the formula

Approximation expression	Triplet (a, b, c)
$\sqrt{\frac{1541+\sqrt{5237}}{\sqrt{2214}}} = \mathbf{5.8556107141}855\ldots \approx 9\sqrt{\pi - e}$	$(1541, 5237, 2214)$
$\sqrt{\frac{20042+\sqrt{4288}}{\frac{5}{2}\sqrt{7359}}} = \mathbf{9.6828710469}359\ldots \approx 4\sqrt{\pi + e}$	$(20042, 4288, 7359)$
$\sqrt{\frac{60095+\sqrt{9042}}{\frac{13}{2}\sqrt{9037}}} = \mathbf{9.869604401}641\ldots \approx \pi^2$	$(60095, 9042, 9037)$
$\sqrt{\frac{214729+\sqrt{7092}}{\frac{43}{2}\sqrt{2093}}} = \mathbf{14.778112197861}267\ldots \approx 2e^2$	$(214729, 7092, 2093)$
$\sqrt{\frac{246910+\sqrt{3977}}{\frac{49}{2}\sqrt{5443}}} = \mathbf{11.6891294612}238\ldots \approx 4\sqrt{\pi e}$	$(246910, 3977, 5443)$
$\sqrt{\frac{352017+\sqrt{6349}}{\frac{71}{2}\sqrt{7038}}} = \mathbf{10.87312731}403\ldots \approx 4e$	$(352017, 6349, 7038)$
$\sqrt{\frac{379814+\sqrt{8656}}{\frac{75}{2}\sqrt{8768}}} = \mathbf{10.4015461481}605\ldots \approx 9\left(\frac{\pi}{e}\right)$	$(379814, 8656, 8768)$
continued ...	

Table 1 ... continued

Approximation expression	Triplet (a, b, c)
$\sqrt{\frac{384129+\sqrt{7316}}{\frac{77}{2}\sqrt{1538}}} = \mathbf{15.952084658}285\ldots \approx 9\sqrt{\pi}$	$(384129, 7316, 1538)$
$\sqrt{\frac{510297+\sqrt{9480}}{\frac{103}{2}\sqrt{228}}} = \mathbf{25.61920266}777\ldots \approx 3\pi e$	$(510297, 9480, 228)$
$\sqrt{\frac{516888+\sqrt{7955}}{\frac{103}{2}\sqrt{4041}}} = \mathbf{12.566370614}406\ldots \approx 4\pi$	$(516888, 7955, 4041)$
$\sqrt{\frac{586602+\sqrt{5746}}{\frac{117}{2}\sqrt{5669}}} = \mathbf{11.541048894}761\ldots \approx 7\sqrt{e}$	$(586602, 5746, 5669)$
$\sqrt{\frac{911862+\sqrt{3403}}{\frac{183}{2}\sqrt{1040}}} = \mathbf{17.579623446}279\ldots \approx 3\,(\pi + e)$	$(911862, 3403, 1040)$

The idea implemented in the computer program to search for the approximations is as follows.

- Check every possible triplet in the range $(1 \leq a \leq 10^6, 1 \leq b \leq 10^4, 1 \leq c \leq 10^4)$ and evaluate the value V according to **Equation (1)**.
- Divide the value V by π, $\sqrt{\pi}$, e, $\sqrt{e}$, $(\pi + e)$, $\sqrt{(\pi + e)}$, π^2, e^2, $\frac{\pi}{e}$, $\sqrt{\frac{\pi}{e}}$, πe, $\sqrt{\pi e}$, $(\pi - e)$, $\sqrt{(\pi - e)}$, $\frac{e}{\pi}$, $\sqrt{\frac{e}{\pi}}$ and find the quotient in each case.
- If any quotient obtained above is an approximate integer (see **Definition 2.1**), then check whether that approximate integer value is a small integer (see **Definition 2.2**). If true, then print that approximation expression.

3. Analysis of results

As mentioned earlier, a total of **158299** approximations have been found by exploring the range defined by the constraints $(1 \leq a \leq 10^6, 1 \leq b \leq 10^4, 1 \leq c \leq 10^4)$. Here are some key insights into the same.

3.1. **Potentially infinite number of triplets.** Looking at the trends of approximations, it appears that there exist potentially infinite number of such approximation formulas for small integer multiples of the specified combinations of π and e. **Figure 1** below shows the number of approximation expressions against the value of a for better understanding.

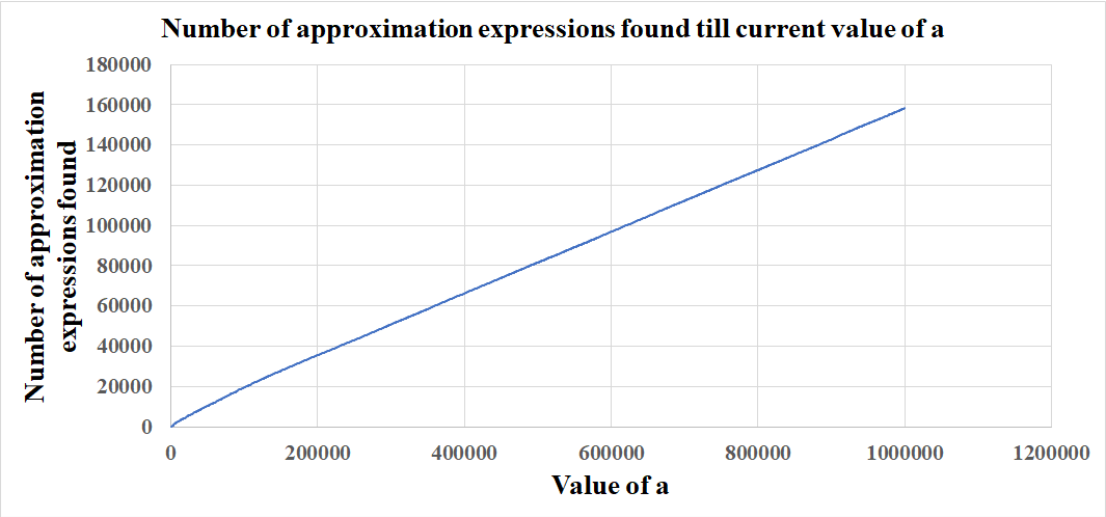

FIGURE 1. Number of approximation expressions vs the value of a

Figure 1 has been plotted by counting the number of approximation expressions found upto a particular value of a. To plot the data, cumulative counts of number of expressions found for every 1000 steps of a value have been taken. For example, there exist 242 approximation expressions upto a = 1000, there are 466 approximation expressions upto a = 2000 and so on.

Clearly, the plotted graph is almost a straight line, and it indicates that *there may exist potentially infinite* number of approximation expressions.

3.2. **Triplet gaps.** Referring to **Definition 2.3** and **Definition 2.4**, the smallest triplet-gap has been observed to be **zero** for the range $(1 \leq a \leq 10^6)$. One such pair of triplets is $(42, 4266, 6455)$ and $(42, 9169, 2116)$, which generate the approximations as: $\sqrt{\frac{42+\sqrt{4266}}{\sqrt{6455}}} \approx \frac{\pi}{e}$ and $\sqrt{\frac{42+\sqrt{9169}}{\sqrt{2116}}} \approx \frac{2e}{\pi}$, respectively.

On the other hand, the largest gap was found as **75** for the range $1 \leq a \leq 10^6$. One such pair of triplets is $(262857, 7699, 7044)$ and $(262932, 2690, 3254)$, and this pair generate the approximations as: $\sqrt{\frac{262857+\sqrt{7699}}{\frac{53}{2}\sqrt{7044}}} \approx 4e$ and $\sqrt{\frac{262932+\sqrt{2690}}{\frac{53}{2}\sqrt{3254}}} \approx 8\sqrt{e}$, respectively.

3.3. **Estimated probability of finding an approximation within a given range of a-values.** An attempt is made to estimate the probability of finding an approximation expression within a given range of a-values. Readers may refer to data in the following table. This is indeed a follow-up to the graph plotted in **Figure 1** in **Section 3.1**. Here are the number of occurrences of these approximation expressions 'a' is increased in steps of 10, 100, 1000, 10000, 100000 over the range $(1 \leq a \leq 10^6)$.

TABLE 2. Number of approximation expressions found per step count increase of a

Step count increase of a	Minimum number of approximation expressions	Maximum number of approximation expressions	Average number of approximation expressions
10	0	17	1.58299
100	3	113	15.8299
1000	106	311	158.299
10000	1446	2531	1582.99
100000	15220	19451	15829.9

Above table is valid for the range ($1 \leq a \leq 10^6$). Looking at the trends for minimum, maximum and average number of approximation expressions as listed above, it can be safely estimated that the probability of finding an approximation expression for any given value of a is $\mathbf{p} \geq \frac{\mathbf{1}}{\mathbf{10}}$.

It means that if a particular value is kept fixed for a and the entire associated range of ($1 \leq b \leq 10^4, 1 \leq c \leq 10^4$) is searched exhaustively, then the probability of finding an approximation of a combination of π and e is given by $\mathbf{p} \geq \frac{\mathbf{1}}{\mathbf{10}}$.

In terms of pure counting, it is estimated that *there exists one approximation among every **1 billion** consecutive triplets* with a high probability.

Another way to visualize these triplets in terms of a three-dimensional coordinate system is illustrated below.

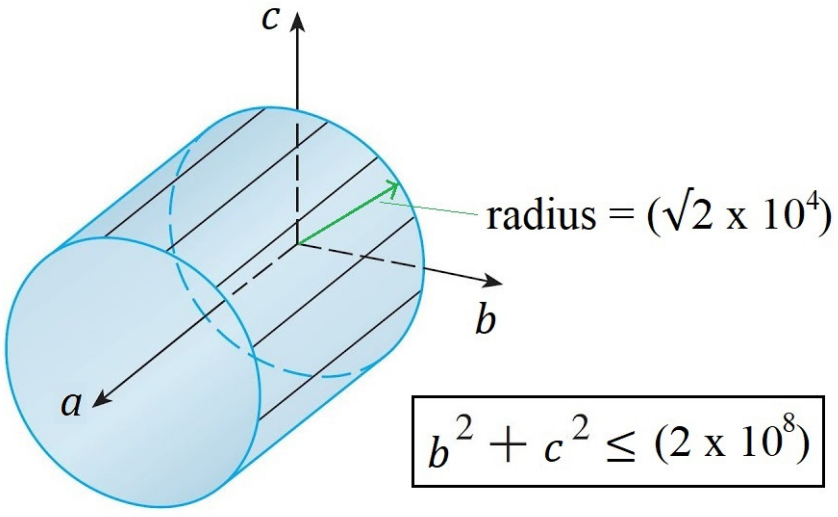

FIGURE 2. Search space for (a, b, c) triplets

According to the constraints ($1 \leq a \leq 10^6, 1 \leq b \leq 10^4, 1 \leq c \leq 10^4$), the triplets

are bounded by a cylinder of radius $(10^4 \times \sqrt{2})$ units. So, the above empirical probability reports that *it is expected to find at least one such approximate expression* in any *disk* of thickness of 10 units along this cylinder.
The aforesaid estimation of the probability is *empirical*, and there is no concrete mathematical proof known for it as of now. No mathematically expressible pattern has been found yet among the triplets generated through these experiments.

3.4. **Distribution of number of approximation expressions.** The number of approximations found for different combinations of π and e for the experimental range $(1 \leq a \leq 10^6, 1 \leq b \leq 10^4, 1 \leq c \leq 10^4)$ are presented below.

Table 3: Number of approximations found for different combinations of π and e

Expression and its **small integer** multiples	Number of approximations
π	12962
$\sqrt{\pi}$	19159
e	22807
$\sqrt{e}$	12998
$(\pi + e)$	25246
$\sqrt{(\pi + e)}$	13244
π^2	6631
e^2	9810
$\frac{\pi}{e}$	9783
$\sqrt{\frac{\pi}{e}}$	3174
πe	5531
$\sqrt{\pi e}$	16653
$(\pi - e)$	12
$\sqrt{(\pi - e)}$	50
$\frac{e}{\pi}$	103
$\sqrt{\frac{e}{\pi}}$	136
Total	**158299**

The same data from **Table 3** above is presented in the graph below for ease of visualization.

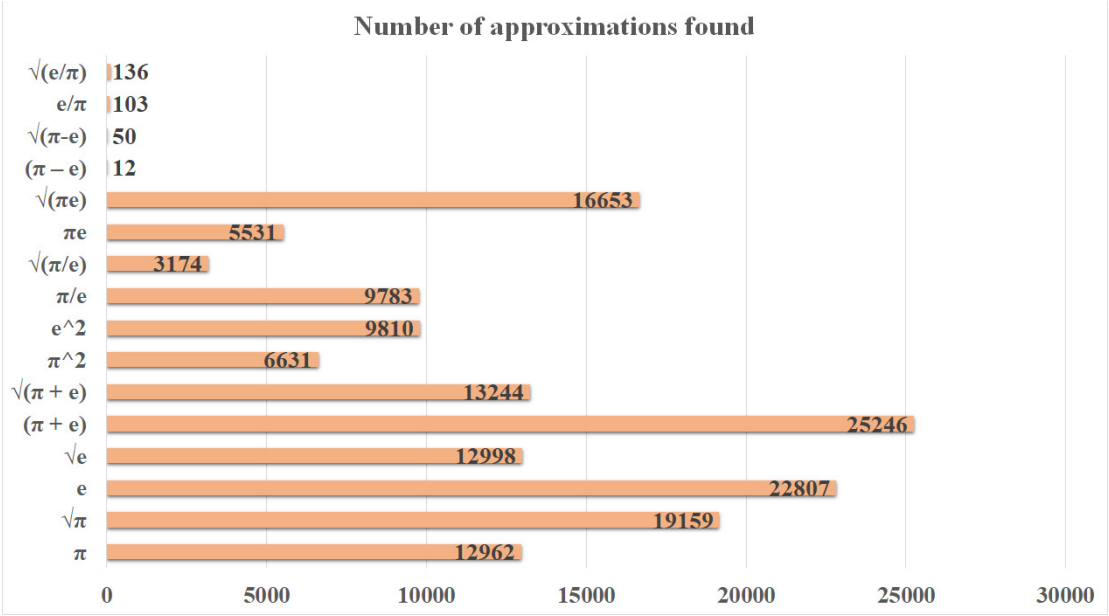

FIGURE 3. Number of approximations found for small integer multiples of different combinations of π and e

3.5. **Repeating patterns for (b, c) part of the triplet (a, b, c).** As defined in **Equation (1)** in **Section 2**, a is left unbounded and the values of b and c are restricted in the range $(1 \leq b \leq 10^4,\ 1 \leq c \leq 10^4)$. So, the (b, c) pair is likely to repeat. However, for these experiments, very-very low repetitions of (b, c) pairs were found when approximation expressions were searched in the range $(1 \leq a \leq 10^6)$.
In particular, out of 158299 approximations discovered, there were just 358 cases of repetition of (b, c) part of the triplet, which implies just 0.226% empirical probability of repetition of (b, c) pairs.

3.6. **Some more mathematical properties of the triplets (a, b, c).**

- Out of the 158299 triplets, there are 138489 triplets or nearly 87.49% which contain *at least one odd number*.
- As there are odd numbers involved, so further investigation revealed that 45879 triplets or nearly 28.98% contain *at least one prime number*.
- $GCD(a, b, c)$ was calculated for the triplets, which revealed few more patterns. A total of 131647 triplets had the GCD value of 1, which accounts for 83.16% of the total triplets found. The maximum GCD value was found to be 299.

  The following graph shows the distribution of number of triplets having a specific GCD.

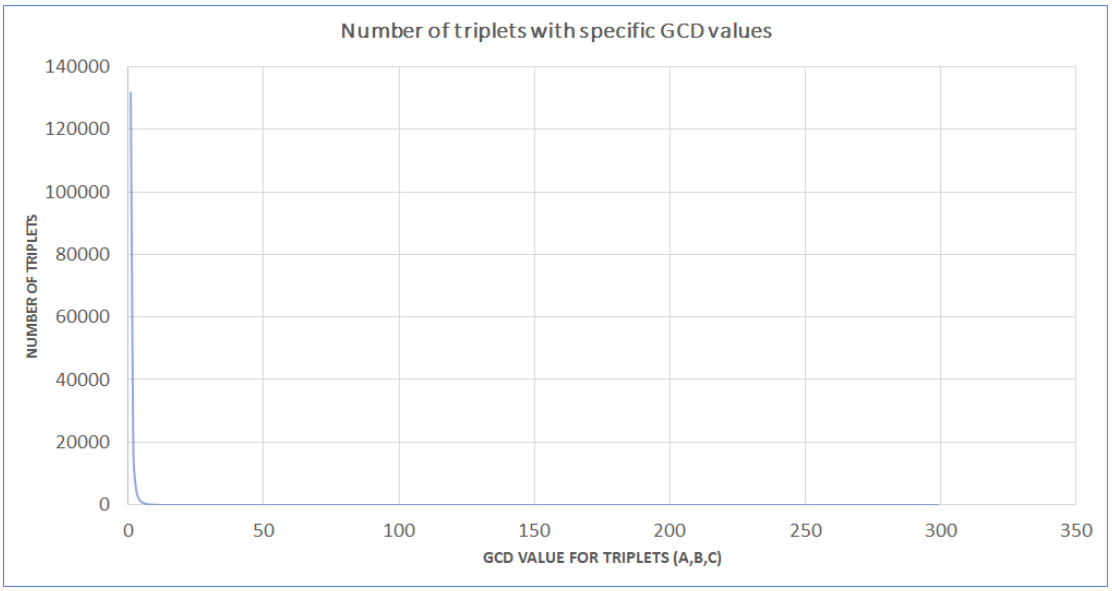

FIGURE 4. Number of (a, b, c) triplets with a specific GCD

The best-fit curve for this data is approximated by $y = \frac{131647}{x^3}$ where y is the number of triplets found with GCD value of x, and $1 \leq x \leq 299$.

4. Significance of this work

The present work has explored an entirely new aspect of representing approximations of multiple combinations of π and e, using *one single mathematical formula*. Another way of looking in this work in context of **Equation (1)** from **Section 2** is that there is a potential to generate possibly infinite sequence of triplets of the form (a, b, c), which can be used as a *secure random number generator*. Random numbers can have applications in several areas like cryptography for data encryption including digital communication for securing data and financial transaction, fast hashing techniques for searching in databases and potentially many other areas.

5. Open Research Problems

Finally, the authors propose five open problems as a by-product of this research. The authors do not have a proof for any of these problems at the time of publication of this article, and will require very deep research from the perspective of number theory.

(1) Are there infinite number of (a, b, c) triplets which generate an approximation of some combination of π and e, correct to at least 8 places of decimals?
(2) Given one (a, b, c) triplet – how to **efficiently** generate the next such approximation triplet? This problem appears to be a potential NP-class problem along the lines of *efficiently generating the next prime number*.
(3) What is the exact probability of getting at least one such (a, b, c) triplet between any two different values of a ?

(4) If the integers from the triplet (a, b, c) are listed in a concatenated way, e.g., if $(4, 7297, 4482)$ is written as a single number **4**7297**4482**, then does listing down all such (a, b, c) triplets in this form generate a secure unpredictable random number generator?
Or a slightly modified version – if only the (b, c) part is listed down as a concatenated number e.g. if the triplet $(4, 7297, 4482)$ is expressed as a single number **7297**4482, then does listing down all such (a, b, c) triplets in this form generate a secure unpredictable random number generator?
If the answer to either of these questions is *Yes*, then this opens a new direction in the fields of encryption and hashing.

(5) It has been found that these (a, b, c) triplets generate approximations of only the mentioned 16 combinations of π and e, and not for other simple irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ etc. What is the reason that these triplets generate approximations only for some specific combinations of π and e ?

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