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On the Properties of Quadrilaterals determined by Triangle Centers

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Abstract. The diagonals of a convex quadrilateral determine four triangles. Within each such triangle select a triangle center (e. g. incenter, centroid, circumcenter, orthocenter, etc.)

These four centers define a new quadrilateral. Is there something interesting that can be said about this quadrilateral? What if the initial quadrilateral has some particular property (e. g. it could have been cyclic, tangential, orthodiagonal, equidiagonal etc.) Does that property transfer to the central quadrilateral? Or maybe the central quadrilateral enjoys some other property?

This type of problem has been considered before, with origins that could be traced as early as 1850's. However, until very recently the research was limited to the case when the centers were among the ones every high school student is familiar with: incenter, centroid, circumcenter, and orthocenter.

In 1993, Clark Kimberling extended the notion of triangle center to include a much wider class of points which can be associated with a triangle. There are currently more than 61,500 such entries in the online Encyclopedia of Triangle Centers (ETC).

In a recent paper which appeared in this journal, Rabinowitz and Suppa investigated the first 1000 centers listed in ETC in the context of the original quadrilateral question and discovered a wealth of new results. In this paper we present simpler proofs of some of their statements and discover some new results by considering different triangle centers.

For example, one of our results is the following:

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Let $ABCD$ be a convex quadrilateral. Let O_1, O_2, O_3, and O_4 be the circumcenters the four triangles BCD, CDA, DAB, and ABC, respectively. Likewise, let N_1, N_2, N_3, and N_4 let the 9-point centers of the same triangles. Then $O_1O_2O_3O_4$ and $N_1N_2N_3N_4$ are similar convex quadrilaterals with opposite orientations.

Keywords. triangle centers, quadrilaterals, barycentric coordinates, analytic Euclidean geometry, computer-discovered mathematics.

Mathematics Subject Classification (2020). 51M04, 51N20, 51-08.

1. Introduction

Euclidean geometry abounds with results which are visually striking but unfortunately, not widely known. Here are a few morsels.

Theorem 1.1. *(a) Let $A_1A_2A_3A_4$ be a cyclic quadrilateral and let H_1, H_2, H_3, H_4 be the orthocenters of triangles $A_2A_3A_4, A_3A_4A_1, A_4A_1A_2$, and $A_1A_2A_3$, respectively. Then $H_1H_2H_3H_4$ is a cyclic quadrilateral as well.*

(b) Let $A_1A_2A_3A_4$ be a cyclic quadrilateral and let I_1, I_2, I_3, I_4 be the incenters of triangles $A_2A_3A_4, A_3A_4A_1, A_4A_1A_2$, and $A_1A_2A_3$, respectively. Then $I_1I_2I_3I_4$ is a rectangle.

(c) Let $A_1A_2A_3A_4$ be a tangential quadrilateral and let O be the intersection of the diagonals A_1A_3 and A_2A_4. Then the incenters I_1, I_2, I_3, I_4 of triangles OA_1A_2, OA_2A_3, OA_3A_4 and OA_4A_1 form a cyclic quadrilateral.

(d) Let $A_1A_2A_3A_4$ be a convex quadrilateral and let O be the intersection of the diagonals A_1A_3 and A_2A_4. Then the centroids G_1, G_2, G_3, G_4 of triangles OA_1A_2, OA_2A_3, OA_3A_4 and OA_4A_1 form a parallelogram.

We could trace the first result to the 1867 book by Griffiths [6] although it was probably known earlier. The second statement is a famous Sangaku problem known as the Japanese theorem for cyclic quadrilaterals. It appears as problem #3.5 in the book of Fukagawa and Pedoe [5]; according to the authors, it was written on a 1880 tablet in the Yamagata prefecture.

Quite surprisingly, result (c) is much more recent. It was discovered experimentally by Michael de Villiers [4] sometime in the mid 1990's. Since de Villiers could not come up with a geometric proof, he asked for help some fellow geometers; eventually, Jordan Tabov and independently, John Wilker solved the problem. De Villiers believes his result is new; he actually mentions the fact that Coxeter himself admitted not to have seen it before.

Result (d) appears in a 2008 paper of Mammana et. al [12] and a generalized version was proved in 2022 by Rabinowitz and Suppa [13].

There is a certain common theme of the results above. The starting point is a reference quadrilateral $A_1A_2A_3A_4$ which determines four triangles (in two different ways). On one hand, one can consider triangles determined by three of the vertices of the reference quadrilateral; this is the case with examples (a) and (b) above. Following the terminology from Rabinowitz and Suppa [13] we refer to this triangles as *half triangles*. On the other hand, one can study the four nonoverlapping triangles determined by parts of the diagonals and sides of the initial

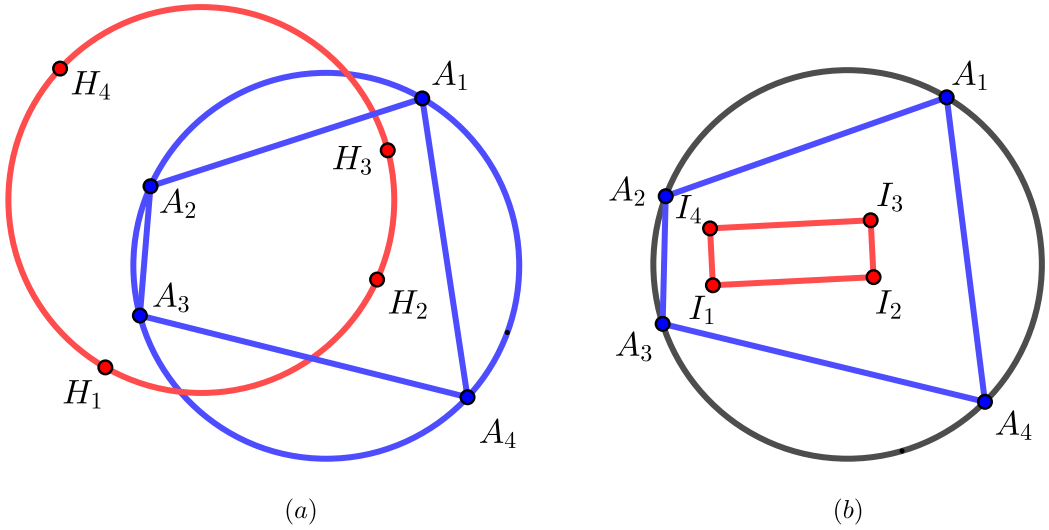

FIGURE 1. (*a*) $A_1A_2A_3A_4$ cyclic $\longrightarrow$ $H_1H_2H_3H_4$ cyclic.
(*b*) $A_1A_2A_3A_4$ cyclic $\longrightarrow$ $I_1I_2I_3I_4$ rectangle.

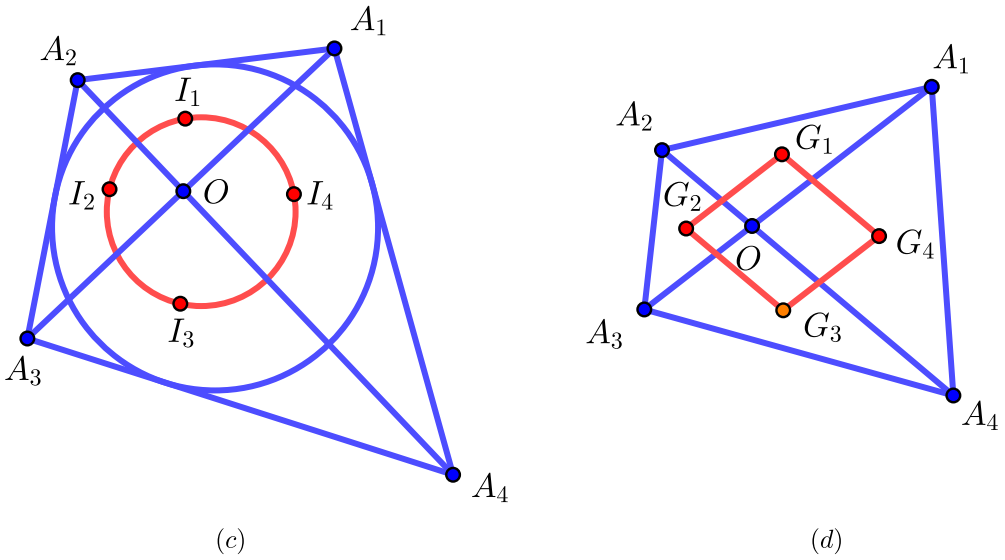

FIGURE 2. (*c*) $A_1A_2A_3A_4$ tangential $\longrightarrow$ $I_1I_2I_3I_4$ cyclic.
(*d*) $A_1A_2A_3A_4$ convex $\longrightarrow$ $G_1G_2G_3G_4$ parallelogram.

quadrilateral. Statements (c) and (d) above fall into this category. We call these triangles *quarter triangles*.

While the term *triangle center* may seem to have a rather limited scope (for instance, most people know of incenter, centroid, circumcenter, orthocenter, and in some cases, 9-point center), nothing can be further from the truth. In fact, there are infinitely many triangle centers, and there exists an online database called the *Encyclopedia of Triangle Centers* (ETC) which currently lists more that 60,000 centers that can be associated to any given triangle.

In 2022, Rabinowitz and Suppa [13] asked whether similar results to those listed in Theorem 1.1 above can be obtained if one replaces the well-known centers (incenter, centroid, circumcenter, orthocenter etc.) by some more exotic ones. They searched for such results among the first 1000 triangle centers listed in ETC and found many new instances in which the quadrilateral determined by the centers has some special property. Rabinowitz and Suppa start with a numerical search and then confirm their findings rigorously by means of barycentric coordinates computations.

Our study has two intended purposes. First, we independently verify some of Rabinowitz and Suppa's results; occasionally, we come up with shorter, cleaner proofs. Second, we extend their search to include triangle centers which they have not considered, or simply missed. We make extensive use of the algebra software MAPLE for (1) discovering the various results and (2) performing the symbolic computations required for the proofs.

2. Barycentric Coordinates and Triangle Centers

For a given triangle ABC, the position of any point X in the same plane as ABC is completely determined by the ratios of the oriented areas of triangle XBC, XCA and XAB to the oriented area of triangle ABC. In vector notation we have

$$\overrightarrow{X} = \frac{Area(XBC)}{Area(ABC)}\overrightarrow{A} + \frac{Area(XCA)}{Area(ABC)}\overrightarrow{B} + \frac{Area(XAB)}{Area(ABC)}\overrightarrow{C}. \tag{1}$$

Here we make the convention that the area of a triangle MNP is zero if M, N and P are collinear, positive if the orientation of the vertices is counterclockwise and negative, otherwise. It follows that the ratios which appear as coefficients in the right hand side of (1) can serve as coordinates of X: we are going to refer to these ratios as the *areal coordinates* of X with respect to the reference triangle ABC - see [3].

One can then write $X = (w_1, w_2, w_3)$ or simply $X = w_1 : w_2 : w_3$, where

$$w_1 = \frac{Area(XBC)}{Area(ABC)},\ w_2 = \frac{Area(XCA)}{Area(ABC)},\ \text{and}\ w_3 = \frac{Area(XAB)}{Area(ABC)}.$$

Note that $w_1 + w_2 + w_3 = 1$. It is often convenient to ignore this side relation and present the coordinates of X in the form

$$X = tw_1 : tw_2 : tw_3, \tag{2}$$

where $t = 0$ can be chosen arbitrarily. These are the *barycentric coordinates* of X. According to [14], Möbius was the first to use barycentric coordinates in 1827. For example the areal coordinates of the centroid G of triangle ABC are $(1/3, 1/3, 1/3)$, but we can simply write $G = 1 : 1 : 1$.

Similarly, if ABC has sidelengths $\|BC\| = a$, $\|CA\| = b$ and $\|AB\| = c$ and I is the incenter of ABC then in terms of areal coordinates

$$I = \frac{r}{h_a} : \frac{r}{h_b} : \frac{r}{h_c} = \frac{a}{a+b+c} : \frac{b}{a+b+c} : \frac{c}{a+b+c}$$

where r is the radius of the incircle and h_a, h_b, h_c are the lengths of the altitudes. Alternatively, we can just write $I = a : b : c$ for a cleaner equivalent output.

In general, a *triangle center function* is a nonzero function f of three real variables a, b and c with the following properties:

- homogeneity: $f(ta, tb, tc) = t^n f(a, b, c)$ for some constant n and all $t > 0$.
- bisymmetry in the second and third variables: $f(a, b, c) = f(a, c, b)$.

If f is a triangle center function and a, b and c are the side lengths of a reference triangle ABC then the point X whose barycentric coordinates are $f(a, b, c) : f(b, c, a) : f(c, a, b)$ is called a *triangle center*.

For example, if O is the circumcenter of the triangle ABC, the barycentric coordinates of O can be written as

$$O = a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2)$$

Similarly, if H denotes the orthocenter of triangle ABC, the barycentric coordinates of H are

$$H = a^4 - (b^2 - c^2)^2 : b^4 - (c^2 - a^2)^2 : (c^4 - (a^2 - b^2)^2.$$

Most of the time it is convenient to specify only the first of the barycentric coordinates of a triangle center since the other two are obtained by cyclic permutations. For instance, if H denotes the orthocenter of triangle ABC one can simply say that its center function is $f(a, b, c) = at-(b^2-c^2)^2$. Every triangle center function corresponds to a unique triangle center. This correspondence is not one-to-one as different functions may define the same triangle center. For example the functions $f(a, b, c) = a$ and $g(a, b, c) = a^2bc$ both correspond to the incenter. Two triangle center functions define the same triangle center if and only if their ratio is a function symmetric in a, b and c. In practice, every triangle center function's domain is restricted to the subset of $\mathbb{R}^3$ where $a < b + c$, $b < c + a$, $c < a + b$ with a, b and c all positive.

To avoid technical complications we are going to restrict ourselves to two types of triangle centers. A triangle center is called a *polynomial triangle center* if its barycentric coordinates can be expressed as polynomials in a, b and c. A triangle center is said to be a *regular triangle center* if its barycentric coordinates can be written as polynomials in a, b c, and S, where S is twice the area of the triangle. Obviously, every polynomial center is also regular. For example, the centroid, incenter, circumcenter, and orthocenter are all polynomial centers.

An example of regular center which is not polynomial is X_{485}, the first (outer) Vecten point of ABC. This triangle center is obtained by constructing the isosceles right triangles $BA'C$, $CB'A$ and $AC'B$ outwardly along the sides BC, CA and AB of triangle ABC. The lines AA', BB', CC' concur in X_{485} -see figure $3(a)$ below.

The center function of this triangle center is given by $f_{485}(a, b, c) = a^2(b^2 + c^2) - (b^2 - c^2)^2 + 2a^2S$. In a similar fashion one can define the second (inner) Vector point of ABC by constructing the isosceles right triangles $BA'C$, $CB'A$ and $AC'B$ inwardly. Again, lines AA', BB', CC' concur in X_{486} - see figure $3(b)$. The center function of the second Vecten point is very similar looking to that of the first Vecten point: $f_{486}(a, b, c) = a^2(b^2 + c^2) - (b^2 - c^2)^2 - 2a^2S$.

As of February 2024, Clark Kimberling's Encyclopedia of Triangle Centers [9] contains an annotated list of over $61,500$ triangle centers.

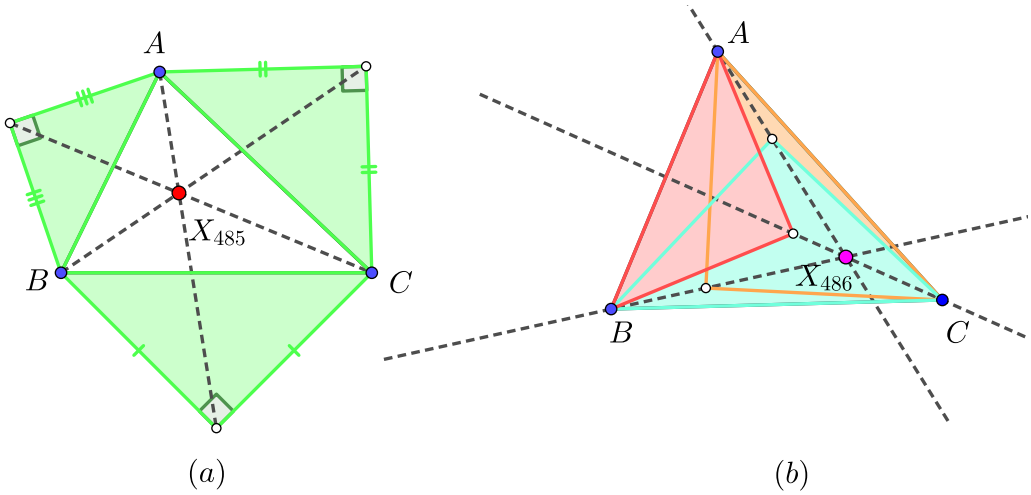

Figure 3. (*a*) The outer Vecten point, X_{485}.
(*b*) The inner Vecten point, X_{486}

3. The coordinates of the reference quadrilateral

As the coordinates of various triangle centers depend on the coordinates of the reference quadrilateral, the choice of the latter is crucial. To this end, we use the following setup.

Given a convex quadrilateral $A_1A_2A_3A_4$, denote by O the intersection point of the diagonals A_1A_3 and A_2A_4. Denote the non-obtuse angle between the diagonals by θ and the lengths of the segments OA_i by r_i for all $1 \le i \le 4$. Place O at the origin and A_1, A_3 along the x-axis.

It follows that the coordinates of the vertices of the reference quadrilateral are given by

$$\begin{aligned} &A_1 = (r_1, 0),\ A_2 = (r_2\cos\theta, r_2\sin\theta),\\ &A_3 = (-r_3, 0),\ \ A_4 = (-r_4\cos\theta, -r_4\sin\theta). \end{aligned} \tag{3}$$

Denote the lengths of the sides A_1A_2, A_2A_3, A_3A_4, and A_4A_1 by s_1, s_2, s_3, and s_4, respectively. Of course, these can be expressed in terms of the r_i-s and the angle θ via the law of cosines.

$$\begin{aligned} &s_1^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta,\ s_2^2 = r_2^2 + r_3^2 + 2r_2r_3\cos\theta,\\ &s_3^2 = r_3^2 + r_4^2 - 2r_3r_4\cos\theta,\ s_4^2 = r_4^2 + r_1^2 + 2r_4r_1\cos\theta. \end{aligned}$$

Finally, denote by $S_1 = 2[OA_1A_2], S_2 = 2[OA_2A_3], S_3 = 2[OA_3A_4]$, and $S_4 = 2[OA_4A_1]$, where $[ABC]$ stands for the area of triangle ABC. Obviously,

$$S_1 = r_1r_2\sin\theta, S_2 = r_2r_3\sin\theta, S_3 = r_3r_4\sin\theta, S_4 = r_4r_1\sin\theta. \tag{4}$$

Note that various additional restrictions on $A_1A_2A_3A_4$ can be easily dealt with. For example, asking of $A_1A_2A_3A_4$ be cyclic, equidiagonal, or orthodiagonal is equivalent to $r_1r_3 = r_2r_4, r_1 + r_3 = r_2 + r_4$, or $\theta = 90°$, respectively.

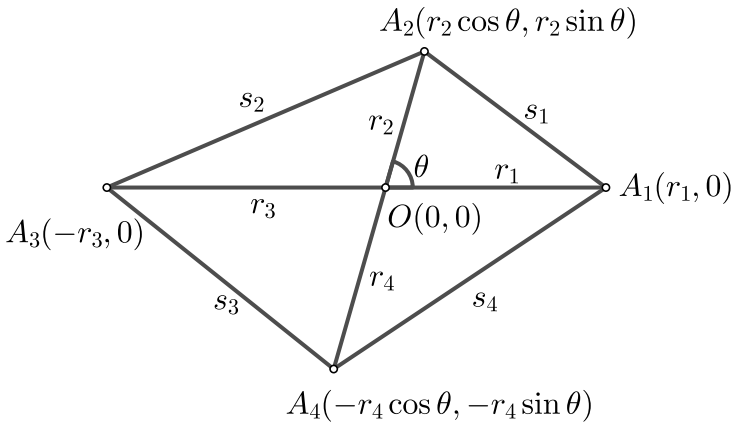

FIGURE 4. The coordinates of the reference quadrilateral

4. **Results for quarter triangles**

It is well known that for any given triangle ABC, the circumcenter, O, the orthocenter, H, the centroid, G, and the 9-point center, N, are collinear. Moreover, N is the midpoint of OH and $\overrightarrow{OG}/\overrightarrow{OH} = 1/3$.

For every real λ, let us consider the point P which divides the segment OH in the ratio $\lambda : 1-\lambda$. In other words, P is defined by the equation $\overrightarrow{OP} : \overrightarrow{PH} = \lambda : 1-\lambda$ where the arrows signify directed segments.

Clearly, P is also a triangle center whose cartesian coordinates can be easily expressed in terms of those of O and H as follows: $P = (1-\lambda)O + \lambda H$.

Theorem 4.1. *Let $A_1A_2A_3A_4$ be a general reference convex quadrilateral whose diagonals intersect at O. Denote by O_1, O_2, O_3, and O_4 be the circumcenters of triangles $OA_1A_2, OA_2A_3, OA_3A_4$, and OA_4A_1, respectively. Let H_1, H_2, H_3, H_4 be the orthocenters of the same triangles. For any real number λ, define $P_i = (1-\lambda)O_i + \lambda H_i$. Then the following statements are true.*

(a) For every λ, the quadrilateral $P_1P_2P_3P_4$ is a parallelogram.

(b) If the reference quadrilateral is equidiagonal ($A_1A_3 = A_2A_4$) then $P_1P_2P_3P_4$ is a rhombus.

(c) If the reference quadrilateral is orthodiagonal ($A_1A_3 \perp A_2A_4$) then $P_1P_2P_3P_4$ is a rectangle.

(d) If the reference quadrilateral $A_1A_2A_3A_4$ is not orthodiagonal then $O_1O_2O_3O_4$ and $H_1H_2H_3H_4$ are similar parallelograms.

Proof. Parts (a), (b) and (c) were proved in [13]. We present alternate proofs below. As far as we know, part (d) is new.

Assume that the coordinates of the vertices of the reference quadrilateral are as in (3)- see figure 4. Using the center function of the circumcenter, $f_3(a,b,c) = a^2(b^2+c^2-a^2)$ one can compute the cartesian coordinates of the circumcenters,

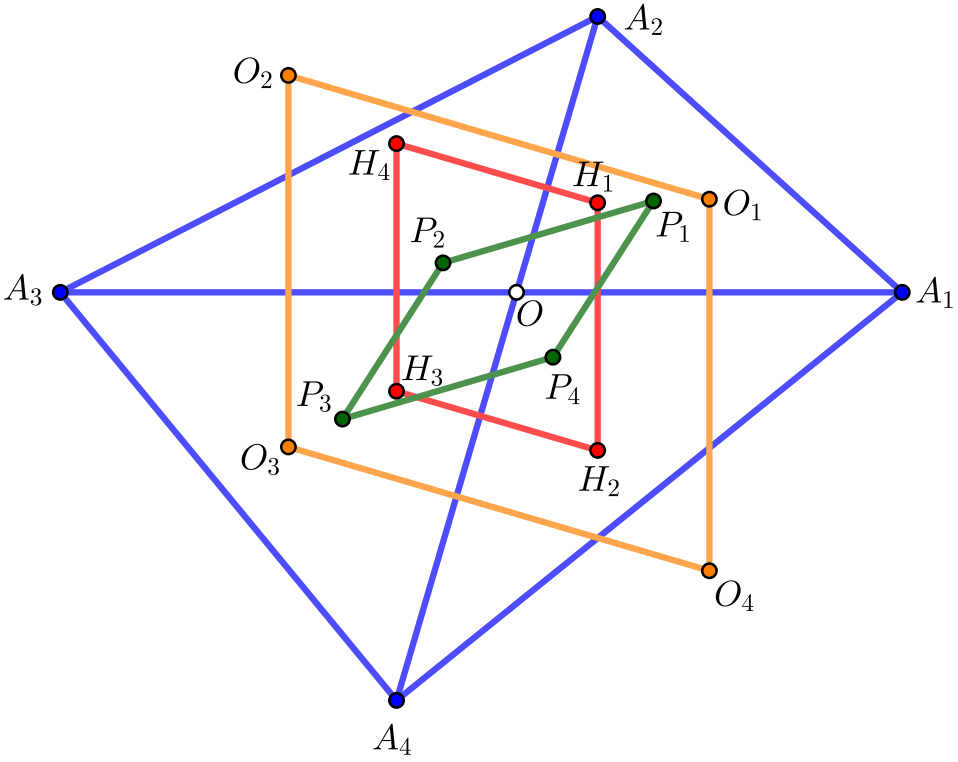

FIGURE 5. $P_i = (1-\lambda)O_i + \lambda H_i \implies P_1P_2P_3P_4$ parallelogram.

O_i, $1 \le i < j \le 4$.

$$O_1 = \left(\frac{r_1}{2}, \frac{r_2 - r_1\cos\theta}{2\sin\theta}\right), \quad O_2 = \left(-\frac{r_3}{2}, \frac{r_2 + r_3\cos\theta}{2\sin\theta}\right),$$
$$O_3 = \left(-\frac{r_3}{2}, \frac{-r_4 + r_3\cos\theta}{2\sin\theta}\right), \quad O_4 = \left(\frac{r_1}{2}, \frac{-r_4 - r_1\cos\theta}{2\sin\theta}\right).$$

It is immediate that $O_1 + O_3 = O_2 + O_4$, hence $O_1O_2O_3O_4$ is a parallelogram.

Similarly, using the center function of the orthocenter, $f_4(a,b,c) = a^4 - (b^2-c^2)^2$ we compute the cartesian coordinates of the orthocenters, H_i, $1 \le i < j \le 4$.

$$H_1 = \left(r_2\cos\theta, \frac{r_1 - r_2\cos\theta}{\sin\theta}\right), H_2 = \left(r_2\cos\theta, \frac{-r_3 - r_2\cos\theta}{\sin\theta}\right),$$
$$H_3 = \left(-r_4\cos\theta, \frac{-r_3 + r_4\cos\theta}{\sin\theta}\right), H_4 = \left(-r_4\cos\theta, \frac{r_1 + r_4\cos\theta}{\sin\theta}\right).$$

Again, it is apparent that $H_1 + H_3 = H_2 + H_4$, hence $H_1H_2H_3H_4$ is also a parallelogram. But since $P_i = (1-\lambda)O_i + \lambda H_i$ for every $1 \le i \le 4$ it immediately follows that $P_1 + P_3 = P_2 + P_4$, which proves that $P_1P_2P_3P_4$ is a parallelogram as well. This proves part (a).

Using the coordinates of the centers P_i, one obtains that

$$P_1P_2^2 = \frac{\lambda^2(1+8\cos^2\theta) - 2\lambda(1+2\cos^2\theta) + (\lambda-1)^2}{4\sin^2\theta} \cdot (r_1+r_3)^2, \text{ and}$$
$$P_2P_3^2 = \frac{\lambda^2(1+8\cos^2\theta) - 2\lambda(1+2\cos^2\theta) + (\lambda-1)^2}{4\sin^2\theta} \cdot (r_2+r_4)^2,$$

which reveals that if $r_1 + r_3 = r_2 + r_4$ then $P_1P_2 = P_2P_3$; this proves part (b).

One can also verify that

$$P_2P_4^2 - P_1P_3^2 = \frac{\cos\theta}{\sin^2\theta}\left((5 + 4\cos^2\theta)\lambda^2 - 6\lambda + 1\right)(r_1 + r_3)(r_2 + r_4),$$

Hence, if $\theta = 90^\circ$ then $P_1P_3 = P_2P_4$, and therefore the parallelogram $P_1P_2P_3P_4$ is a rectangle, thus proving part (c).

Straightforward calculations show that

$$O_1O_2 = \frac{r_1 + r_3}{2\sin\theta}, \text{ and } H_1H_2 = (r_1 + r_3)\cot\theta,$$

from which $H_1H_2/O_1O_2 = 2\cos\theta$, and the same value is reached by every quotient of the form H_iH_j/O_iO_j, $1 \leq i < j \leq 4$. Hence, $H_1H_2H_2H_4$ and $O_1O_2O_3O_4$ are similar parallelograms with proportionality ratio $2\cos\theta$. Obviously, we must exclude the case when $\theta = 90^\circ$ as in this situation all four orthocenters coincide. Part (d) is thus complete.

Finally, let us consider the situation when $P_1P_2P_3P_4$ is degenerate. It turns out that

$$\begin{vmatrix} 1 & x_{P_1} & y_{P_1} \\ 1 & x_{P_2} & y_{P_2} \\ 1 & x_{P_3} & y_{P_3} \end{vmatrix} = \frac{(\lambda - 1 - 2\lambda\cos\theta)(\lambda - 1 + 2\lambda\cos\theta)}{4\sin\theta}(r_1 + r_3)(r_2 + r_4),$$

and the right hand side in the equality above vanishes when $\lambda - 1 = \pm 2\lambda\cos\theta$.

For example, if $\lambda = 1$ and $\theta = 90^\circ$ we obtain the familiar case of coinciding orthocenters if the reference quadrilateral is orthodiagonal. A more interesting example is when $\lambda = 1/2$ and $\theta = 60^\circ$. In this case, the centers $P_i = (O_i + H_i)/2$ are the 9-point centers of the four quarter triangles and they are collinear. □

The inspiration for the next result originates in the paper of Rabinowitz and Suppa [13].

Theorem 4.2. [13] *Let $A_1A_2A_3A_4$ be a convex cyclic quadrilateral with diagonal point O and let P_1, P_2, P_3, P_4 be the centers of triangles OA_1A_2, OA_2A_3, OA_3A_4, and OA_4A_1, respectively. Then if the center is one of the Vecten points (X_{485} or X_{486}) then $P_1P_3 \perp P_2P_4$.*

We extend this result in two ways. On one hand, we prove that each of the Kenmotu points, X_{371} or X_{372}, has the same property. On the other hand, the result holds not only when the reference quadrilateral is cyclic, but also when it is orthodiagonal. Also, in the case of Vecten points, the result holds if the angle between the diagonals of the reference quadrilateral is 45°.

The *first Kenmotu point* is the triangle center constructed by inscribing three equal isosceles right triangles such that each triangle touches two sides and all three triangles share the right angle vertex - see figure $6(a)$. The first Kenmotu point is the Kimberling center X_{371} and its triangle center function is $f_{371}(a, b, c) = a^2(b^2 + c^2 - a^2 + 2S)$. Recall that $S =$ twice the area of triangle ABC. Similarly, one can inscribe three congruent isosceles right triangles which share the right angle vertex and each triangle touches the extensions of two sides - see figure $6(b)$. Thus we obtain the second Kenmotu point, X_{372}, whose center function is $f_{372}(a, b, c) = a^2(b^2 + c^2 - a^2 - 2S)$.

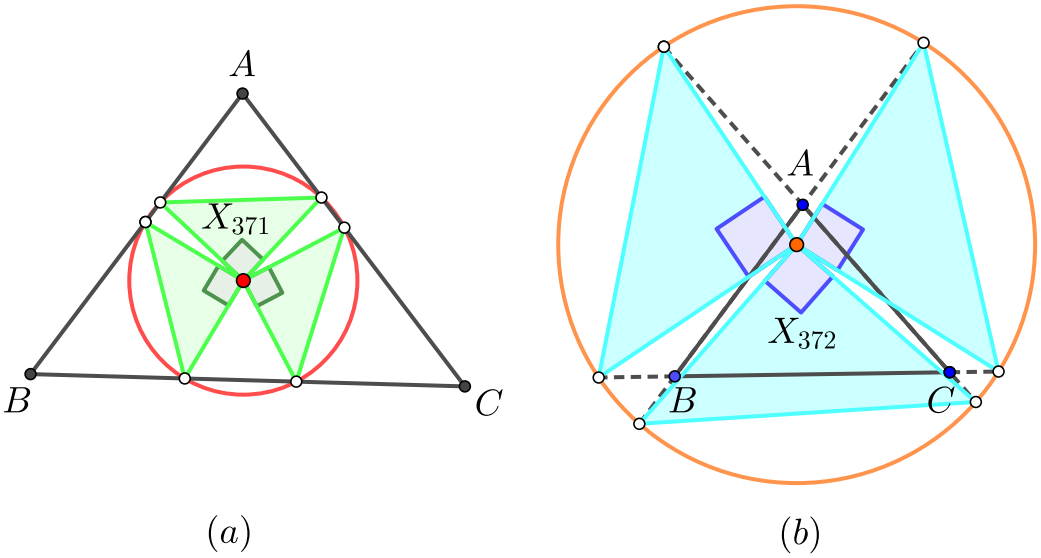

FIGURE 6. (*a*) The first (inner) Kenmotu point, X_{371}.
(*b*) The second (outer) Kenmotu point, X_{372}

Given a point M in the plane of the triangle ABC reflect the lines AM, BM, and CM about the interior angle bisectors at A, B, and C, respectively. One can show that the three reflected lines concur at a point N, which is called *the isogonal conjugate* of M.

It may be worth noting that the first Vecten point, X_{485}, and the first Kenmotu point, X_{371}, are isogonal conjugates of each other. The same relationship holds between the second Vecten point, X_{486}, and the second Kenmotu point, X_{372}.

Here is the extended version of Theorem 4.2 promised earlier.

Theorem 4.3. *Let $A_1A_2A_3A_4$ be a convex quadrilateral with diagonal point O, and denote by θ be the non-obtuse angle between the diagonals A_1A_3 and A_2A_4. Let P_1, P_2, P_3, P_4 be the centers of triangles OA_1A_2, OA_2A_3, OA_3A_4, and OA_4A_1, respectively.*

(a) If $A_1A_2A_3A_4$ is cyclic and the center P is one of the Vecten points or one of the Kenmotu points then $P_1P_3 \perp P_2P_4$.

(b) If $A_1A_2A_3A_4$ is orthodiagonal ($\theta = 90°$), and the center P is one of the Vecten points or one of the Kenmotu points then $P_1P_3 \perp P_2P_4$. Moreover, in this case the centers lie on the angle bisectors of the right angles determined by the diagonals A_1A_3 and A_2A_4.

(c) If the angle θ, formed by the diagonals of a reference convex quadrilateral $A_1A_2A_3A_4$ equals $45°$ and the center P is one of the Vecten points then $P_1P_3 \perp P_2P_4$.

Proof. Let $A_1A_2A_3A_4$ be an arbitrary convex quadrilateral and assume first that $P = X_{485}$, the first Vecten point. Then, by using the center function $f_{485}(a, b, c) = a^2(b^2 + c^2) - (b^2 - c^2)^2 + 2a^2S$ with the coordinates from figure 4 and the obvious substitutions $S(OA_1A_2) = r_1r_2 \sin\theta, S(OA_2A_3) = r_2r_3 \sin\theta \ldots$, we obtain the coordinates of the centers P_1, P_2, P_3, P_4.

For example, the cartesian coordinates of P_1 are

$$P_1 = \left(\frac{r_1 r_2 \left(r_1(\sin\theta + \cos\theta) + r_2(1 + \sin 2\theta)\right)}{2(r_1^2 + r_2^2 + r_1 r_2(2\sin\theta - \cos\theta))}, \frac{r_1 r_2 \left(r_1(\sin\theta + \cos\theta) - r_2 \cos 2\theta\right)}{2(r_1^2 + r_2^2 + r_1 r_2(2\sin\theta - \cos\theta))} \right)$$

The coordinates of the other centers, P_2, P_3, and P_4 look very similar.

Computing the dot product $\overrightarrow{P_1P_3} \cdot \overrightarrow{P_2P_4}$ it turns out that its numerator can be written as

$$\text{numerator}(\overrightarrow{P_1P_3} \cdot \overrightarrow{P_2P_4}) = \cos\theta \cos 2\theta \, (r_1 r_3 - r_2 r_4) \, F(r_1, r_2, r_3, r_4, \theta),$$

where $F(r_1, r_2, r_3, r_4)$ is a homogeneous polynomial of degree 8 in the variables r_1, r_2, r_3, r_4 whose coefficients are linear combinations of $1, \sin\theta$, and $\cos\theta$. It follows that $P_1P_3 \perp P_2P_4$ as soon as $\theta = 90°$, $\theta = 45°$, or $A_1A_2A_3A_4$ is cyclic.

Moreover, if $\theta = 90°$, note that the coordinates of P_1 given above become equal to each other. This means that in this case P_1 lies on the angle bisector of the right angle $\angle A_1OA_2$ and identical results hold for the other three centers. This completes the proof in the case when $P = X_{485}$, the first Vecten point. The scenario $P = X_{486}$ is very similar, hence we choose to omit the details.

Suppose next that $A_1A_2A_3A_4$ is an arbitrary convex reference quadrilateral and $P = X_{371}$, the first Kenmotu point. Using now the center function $f_{371}(a, b, c) = a^2(b^2 + c^2 - a^2 + 2S)$ in each of the triangles A_1OA_2, A_2OA_3, A_3OA_4, and A_4OA_1 we obtain the coordinates of the centers P_1, P_2, P_3, and P_4, respectively.

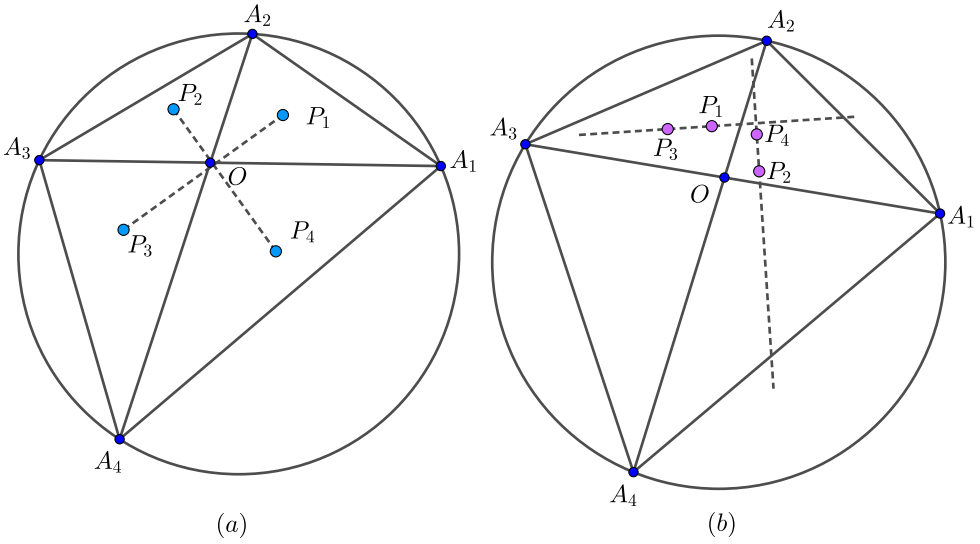

FIGURE 7. (a) $A_1A_2A_3A_4$ cyclic and $P = X_{371}$ then $P_1P_3 \perp P_2P_4$
(b) $A_1A_2A_3A_4$ cyclic and $P = X_{372}$ then $P_1P_3 \perp P_2P_4$

In this case

$$P_1 = \left(\frac{r_1 r_2 (r_1(\sin\theta + \cos\theta) + r_2)}{2(r_1^2 + r_2^2 + r_1 r_2(\sin\theta - \cos\theta))}, \frac{r_1 r_2 (r_1(\sin\theta - \cos\theta) + r_2)}{2(r_1^2 + r_2^2 + r_1 r_2(\sin\theta - \cos\theta))} \right),$$

with similar expressions for the coordinates of the remaining three centers. Similar to the Vecten case, the numerator of the dot product $\overrightarrow{P_1P_3} \cdot \overrightarrow{P_2P_4}$ can be written as

$$\text{numerator}(\overrightarrow{P_1P_3} \cdot \overrightarrow{P_2P_4}) = \cos\theta \, (r_1 r_3 - r_2 r_4) \, G(r_1, r_2, r_3, r_4, \theta),$$

where $G(r_1, r_2, r_3, r_4)$ is a homogeneous polynomial of degree 8 in the variables r_1, r_2, r_3, r_4 whose coefficients are linear combinations of $1, \sin\theta$, and $\cos\theta$.

It follows that $P_1P_3 \perp P_2P_4$ as soon as $\theta = 90°$, or $A_1A_2A_3A_4$ is cyclic. As in the case above, if $\theta = 90°$, the coordinates of P_1 given above are equal to each other, hence P_1 lies on the angle bisector of the right angle $\angle A_1OA_2$ and the same result holds for the other three centers.

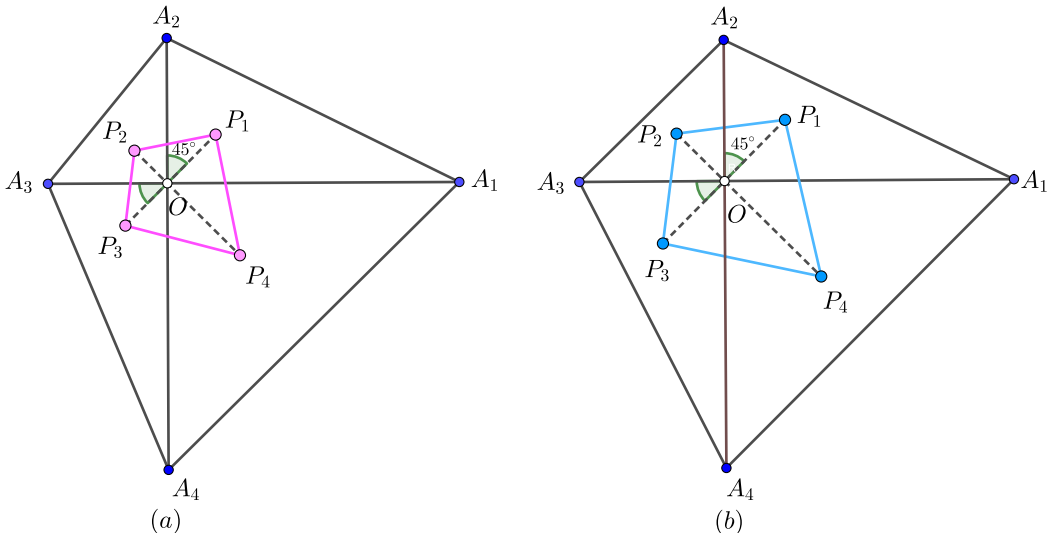

FIGURE 8. (*a*) If $\theta = 90°$, $P = X_{485} \longrightarrow P_1P_3$, P_2P_4 bisect $\angle O$.
(*b*) If $\theta = 90°$, $P = X_{371} \longrightarrow P_1P_3$, P_2P_4 bisect $\angle O$.

□

5. Results for half triangles

While experimenting with Geogebra we noticed an unexpected relation between the central quadrilaterals determined by the circumcenters and the 9-point centers of the half triangles of a convex reference quadrilateral.

Theorem 5.1. *For a given noncyclic convex quadrilateral $A_1A_2A_3A_4$ let O_1, O_2, O_3, O_4 and N_1, N_2, N_3, N_4 be the circumcenters and the 9-point centers of triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. Then the quadrilaterals $O_1O_2O_3O_4$ and $N_1N_2N_3N_4$ are similar and have opposite orientations.*

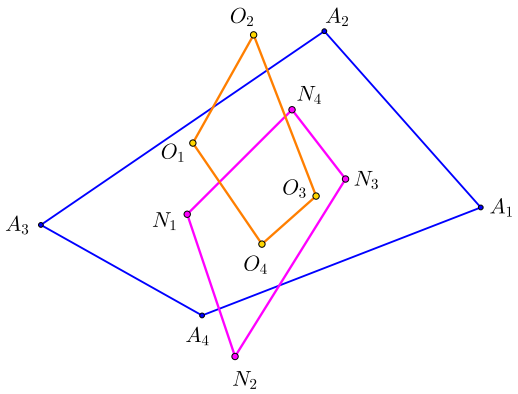

FIGURE 9. $O_1O_2O_3O_4 \simeq N_1N_2N_3N_4$

This is a surprising result for a few reasons. First, it holds for any convex quadrilateral $A_1A_2A_3A_4$; we impose the restriction that not all four points are concyclic since in this case the circumcenters O_1, O_2, O_3, and O_4 would coincide.

Second, the two configurations are not homothetic; in other words, there is no easy similarity transformation that maps O_i into N_i. Third, the similarity ratio is not a constant, but it depends on the initial configuration in a way that we are going to make explicit below.

Proof. Consider the setting described in figure 4. Using the center function of the circumcenter, $f_3(a,b,c) = a^2(b^2+c^2-a^2)$, for each of the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$ we obtain the cartesian coordinates of O_1, O_2, O_3, and O_4. These coordinates are relatively simple as shown below.

$$O_1 = \left(\frac{r_2r_4 - r_3^2}{2r_3},\ \frac{r_3(r_2-r_4)+(r_3^2-r_2r_4)\cos\theta}{2r_3\sin\theta}\right),$$
$$O_2 = \left(\frac{r_1 - r_3}{2},\ \frac{r_1r_3 - r_4^2 + r_4(r_3-r_1)\cos\theta}{2r_4\sin\theta}\right),$$
$$O_3 = \left(\frac{r_1^2 - r_2r_4}{2r_1},\ \frac{r_1(r_2-r_4)+(r_2r_4-r_1^2)\cos\theta}{2r_1\sin\theta}\right),$$
$$O_4 = \left(\frac{r_1 - r_3}{2},\ \frac{r_3(r_2-r_4)+(r_3^2-r_2r_4)\cos\theta}{2r_2\sin\theta}\right).$$

In a similar manner, by using the center function of the 9-point center, $f_5(a,b,c) = a^2(b^2+c^2) - (b^2-c^2)^2$, one can compute the coordinates of N_1, N_2, N_3, and N_4.

$$N_1 = \left(\frac{-r_3^2 - r_2r_4 + 2r_3(r_2-r_4)\cos\theta}{4r_3},\ \frac{(r_2r_4-r_3^2)\cos\theta + r_3(r_4-r_2)\cos 2\theta}{4r_3\sin\theta}\right),$$
$$N_2 = \left(\frac{r_1 - r_3 - 2r_4\cos\theta}{4},\ \frac{-r_1r_3 + r_4(r_1-r_3)\cos\theta + r_4^2\cos 2\theta}{4r_4\sin\theta}\right),$$
$$N_3 = \left(\frac{r_1^2 + r_2r_4 + 2r_1(r_2-r_4)\cos\theta}{4r_1},\ \frac{(r_1^2-r_2r_4)\cos\theta + r_1(r_4-r_2)\cos 2\theta}{4r_1\sin\theta}\right),$$
$$N_4 = \left(\frac{r_1 - r_3 + 2r_2\cos\theta}{4},\ \frac{r_1r_3 + r_2(r_1-r_3)\cos\theta - r_2^2\cos 2\theta}{4r_2\sin\theta}\right).$$

One can now easily calculate the distances O_iO_j and N_iN_j. For example,

$$O_1O_2^2 = \frac{(r_1r_3 - r_2r_4)^2}{4r_3^2r_4^2\sin^2\theta}s_3^2, \text{ and } N_1N_2^2 = \frac{(r_1r_3 - r_2r_4)^2 + 4r_1r_2r_3r_4\sin^2\theta}{16r_3^2r_4^2\sin^2\theta}s_3^2,$$

from which one readily obtains that

$$\frac{O_1O_2^2}{N_1N_2^2} = \frac{4(r_1r_3 - r_2r_4)^2}{(r_1r_3 - r_2r_4)^2 + 4r_1r_2r_3r_4\sin^2\theta}. \tag{5}$$

Note that the right hand term of the equality above is invariant under circular permutations of (r_1, r_2, r_3, r_4). Moreover, it becomes equal to 0 if $r_1r_3 = r_2r_4$, that is, when $A_1A_2A_3A_4$ is concyclic, as expected. One can easily verify that the quotient $O_iO_j^2/N_iN_j^2$ takes the exact same value for all choices $1 \le i < j \le 4$. It follows that quadrilaterals $O_1O_2O_3O_4$ and $N_1N_2N_3N_4$ are similar, as claimed.

Finally, one could notice by looking at figure 9 that $N_1N_2N_3N_4$ is convex and has the same orientation as the reference quadrilateral $A_1A_2A_3A_4$, while $O_1O_2O_3O_4$ is also convex but has the opposite orientation. This is true in general and can be easily verified.

For example, by computing the signed area of triangle $N_1N_2N_3$ we obtain

$$\begin{vmatrix} 1 & x_{N_1} & y_{N_1} \\ 1 & x_{N_2} & y_{N_2} \\ 1 & x_{N_3} & y_{N_3} \end{vmatrix} = \frac{(r_1+r_3)((r_1r_3-r_2r_4)^2+4r_1r_2r_3r_4\sin^2\theta)}{4r_1r_3r_4\sin\theta} > 0,$$

which shows that $N_1N_2N_3$ has counterclockwise orientation. On the other hand, the analogous computation for triangle $O_1O_2O_3$ gives

$$\begin{vmatrix} 1 & x_{O_1} & y_{O_1} \\ 1 & x_{O_2} & y_{O_2} \\ 1 & x_{O_3} & y_{O_3} \end{vmatrix} = -\frac{(r_1+r_3)(r_1r_3-r_2r_4)^2}{4r_1r_3r_4\sin\theta} < 0,$$

which proves the clockwise orientation of $O_1O_2O_3$. Similar computations can be performed for the remaining triangles. □

Our last result involves certain triangle centers that lie on the *Soddy line*. This line passes through Kimberling centers X_i for $i = 1$ (incenter), 7 (Gergonne point), 20 (de Longchamps point), 77, 170, 175 (outer Soddy center), 176 (inner Soddy center S), 269, 279, 347, 390, 481 (first Eppstein point), 482 (second Eppstein point), 962 (the Longuet-Higgins point), 990, 991, 1042, 1044, 1323 (Fletcher point), 1371 (inner Rigby point), 1372 (outer Rigby point), 1373 (inner Griffiths point), 1374 (outer Griffiths point), 1442, 1443, 1448, 1458, 1721, 1742, 1770, 2263, 2293, 2951, 3000, 3007, 3010, 3012, 3019 etc.

Consider the center X_{1770} which can be defined as a convex combination of X_1 and X_{20} as below

$$X_{1770} = (1-\mu)X_1 + \mu X_{20}, \quad \text{where } \mu = \frac{(b+c-a)(c+a-b)(a+b-c)}{2abc}.$$

Theorem 5.2. *For any real k, let $P = X_{1770} + kX_1$ be the corresponding triangle center. Let $A_1A_2A_3A_4$ be a cyclic quadrilateral and denote by P_1, P_2, P_3, P_4 the P-centers of triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. Then, $P_1P_2P_3P_4$ is a parallelogram.*

Proof. In figure 10, the P_i is the X_{1770} center and I_i is the incenter of the half triangle $A_{i+1}A_{i+2}A_{i+3}$. Similarly, Q_i is the midpoint of I_iP_i which turns to be the X_{4292} center of $A_{i+1}A_{i+2}A_{i+3}$. As mentioned earlier, $I_1I_2I_3I_4$ is a rectangle. However, for any $\lambda = 0$, the points R_i defined by $R_i = (1-\lambda)I_i + \lambda P_i$, are the vertices of a non-rectangular parallelogram.

As the center function of X_{1770} is rather complicated,

$$f_{1770}(a,b,c) = 2a^4-(b^4+c^4)+a^3(b+c)-a^2(b^2+c^2)-a(b^3+c^3+b^2c+bc^2)-(b^2-c^2)^2,$$

we use a different setup in order to verify this result.

Without loss of generality we may assume that $A_1(1,0)$, $A_2(\cos 2\alpha, \sin 2\alpha)$, $A_3(\cos 2\beta, \sin 2\beta)$, $A_4(\cos 2\gamma, \sin 2\gamma)$, are the vertices of the initial cyclic quadrilateral with $0 < \alpha < \beta < \gamma < \pi$. Since we want to avoid having to simplify trigonometric expressions we denote $\cot\alpha = t$, $\cot\beta = u$ and $\cot\gamma = v$. Clearly,

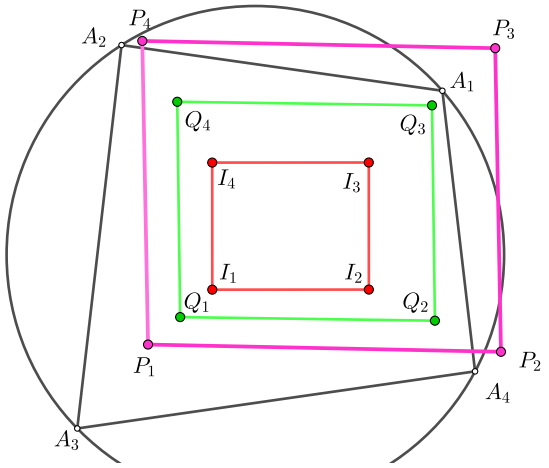

Figure 10.

since the cotangent function is decreasing on $(0,\pi)$ we have that $t>u>v$. It is then easy to see that the coordinates of the points A_i can be written as:

$$A_1(1,0),\quad A_2\left(\frac{t^2-1}{t^2+1},\frac{2t}{t^2+1}\right),\quad A_3\left(\frac{u^2-1}{u^2+1},\frac{2u}{u^2+1}\right),\quad A_4\left(\frac{v^2-1}{v^2+1},\frac{2v}{v^2+1}\right).$$

Straightforward computations allow us to compute the lengths of the segments A_iA_j, $1\le i<j\le 4$,

$$\begin{aligned} &A_1A_2=\frac{2}{\sqrt{t^2+1}},\ A_1A_3=\frac{2}{\sqrt{u^2+1}},\ A_2A_3=\frac{2(t-u)}{\sqrt{t^2+1}\sqrt{u^2+1}},\\ &A_1A_4=\frac{2}{\sqrt{v^2+1}},\ A_2A_4=\frac{2(t-v)}{\sqrt{t^2+1}\sqrt{v^2+1}},\ A_3A_4=\frac{2(u-v)}{\sqrt{u^2+1}\sqrt{v^2+1}}. \end{aligned}\tag{6}$$

This leads to expressions involving the radicals $\sqrt{t^2+1}$, $\sqrt{u^2+1}$ and $\sqrt{v^2+1}$ which appear in (6). The subsequent verification of the claimed parallelogram condition becomes more time consuming, as MAPLE seems to take much longer simplifying an expression containing radicals than a rational one. It is therefore useful to introduce three new variables as explained below.

Let $t>u>v$ be the parameters used to define the points A_2, A_3 and A_4 as shown above. Consider three new positive quantities q, r and s such that

$$t=\frac{1-q^2}{2q},\ u=\frac{1-r^2}{2r},\ v=\frac{1-s^2}{2s}.\tag{7}$$

Note that since the function $g(x)=(1-x^2)/2x$, $g:(0,\infty)\longrightarrow\mathbb{R}$ is decreasing and surjective the choices for q, r and s are unique. Moreover, the condition $t>u>v$ implies that $0<q<r<s$. With these new variables the coordinates of the initial points A_i can be written as,

$$A_1(1,0),\quad A_2\left(\frac{q^4-6q^2+1}{(q^2+1)^2},\frac{4q(1-q^2)}{(q^2+1)^2}\right),\quad A_3\left(\frac{r^4-6r^2+1}{(r^2+1)^2},\frac{4r(1-r^2)}{(r^2+1)^2}\right),$$

and $A_4\left(\dfrac{s^4-6s^2+1}{(s^2+1)^2},\dfrac{4s(1-s^2)}{(s^2+1)^2}\right)$.

The benefit is now that all distances A_iA_j can be written as rational functions as q, r and s.

$$A_1A_2 = \frac{4q}{q^2+1},\ A_1A_3 = \frac{4r}{r^2+1},\ A_2A_3 = \frac{4(r-q)(1+rq)}{(q^2+1)(r^2+1)},$$
$$A_1A_4 = \frac{4s}{s^2+1},\ A_2A_4 = \frac{4(s-q)(1+sq)}{(q^2+1)(s^2+1)},\ A_3A_4 = \frac{4(s-r)(1+sr)}{(r^2+1)(s^2+1)}. \tag{8}$$

It is now easy to employ MAPLE to calculate the coordinates of P_1, P_2, P_3, and P_4, the X_{1770} centers of triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. For example the coordinates of P_4 satisfy the following equalities:

$$\begin{aligned} x_{P_4} =&(q^6r^6 - 17q^6r^4 + 12q^5r^5 + 3q^4r^6 + 11q^6r^2 - 8q^5r^3 - 15q^4r^4 - 8q^3r^5 + \\ &+3q^2r^6 - 3q^6 + 12q^5r + 57q^4r^2 - 80q^3r^3 - 27q^2r^4 + 12qr^5 + r^6 + 11q^4 - \\ &-8q^3r - 15q^2r^2 - 8qr^3 + 3r^4 - 17q^2 + 12qr + 3r^2 + 1)/\left((q^2+1)^3(r^2+1)^3\right), \end{aligned}$$

$$\begin{aligned} y_{P_4} =&4q(1-qr)(q^4r^4 - 6q^4r^2 + 4q^3r^3 + 6q^2r^4 + q^4 - 12q^3r + 20qr^3 - 3r^4 - \\ &-6q^2 + 4qr + 6r^2 + 1)/\left((q^2+1)^3(r^2+1)^3\right), \end{aligned}$$

while P_1, P_2, and P_3 have very similar looking coordinates, all of which are rational functions of q, r, and s.

It is then a matter of computer verification that $P_1 + P_3 = P_2 + P_4$, that is, $P_1P_2P_3P_4$ is indeed a parallelogram, as claimed. □

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