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Four Concentric Circles about the Incenter

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1. Introduction

In [1], we made a certain construction in the context of a triangle and deduced some interesting figures from it. Here we will deduce more figures, specifically four circles which happen to be concentric about the incenter.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be the point on line AC, with $\angle CAB = \angle DAB$, such that $AD = AB$.
Let E be the point on line AB, with $\angle CAB = \angle CAE$, such that $AE = AC$.
Let F be the point on line BC, with $\angle CBA = \angle FBA$, such that $BF = AB$.
Let G be the point on line AB, with $\angle CBA = \angle CBG$, such that $BG = BC$.
Let H be the point on line BC, with $\angle ACB = \angle ACH$, such that $CH = AC$.
Let I be the point on line AC, with $\angle ACB = \angle ICB$, such that $CI = BC$.
Let M be the circumcenter of A, G, and I.
Let N be the circumcenter of C, D, and F.
Let O be the circumcenter of B, E, and H.
Then circles AFH, BDI, CEG, and MNO are concentric about the incenter of triangle ABC.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates $(\frac{x_C}{\sqrt{x_C^2+y_C^2}}, \frac{y_C}{\sqrt{x_C^2+y_C^2}})$.
E has coordinates $(\sqrt{x_C^2+y_C^2}, 0)$.
F has coordinates $(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1, \frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})$.
G has coordinates $(1-\sqrt{(x_C-1)^2+y_C^2}, 0)$.
H has coordinates $(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}, y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))$.
I has coordinates $(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}), y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))$.
M has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
N has coordinates $(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}, -\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})$.
O has coordinates $((1+\sqrt{x_C^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
Consider the circle whose equation is given by
$(x-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(y-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+$
$(\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0.$
We note that this circle is centered at the incenter of triangle ABC, and we verify that points A, F, and H lie on it since we calculate
$(0-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(0-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+$
$(\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$
$(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-$
$((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$
and
$(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-$
$((\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0.$
Now, consider the circle whose equation is given by
$(x-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(y-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((1-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+$
$(\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0.$
We note that this circle is centered at the incenter of triangle ABC, and we verify that points B, D, and I lie on it since we calculate
$(1-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(0-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((1-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+$
$(\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$
$(\frac{x_C}{\sqrt{x_C^2+y_C^2}}-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(\frac{y_C}{\sqrt{x_C^2+y_C^2}}-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((1-$
$\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$

and

$$(x_C * (1 - \frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}) - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y_C * (1 - \frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}) - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((1 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0.$$

Next, consider the circle whose equation is given by

$$(x - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((x_C - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y_C - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0.$$

We note that this circle is centered at the incenter of triangle ABC, and we verify that points C, E, and G lie on it since we calculate

$$(x_C - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y_C - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((x_C - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y_C - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$$

$$(\sqrt{x_C^2+y_C^2} - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (0 - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((x_C - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y_C - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$$

and

$$(1 - \sqrt{(x_C-1)^2+y_C^2} - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (0 - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - ((x_C - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y_C - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0.$$

Finally, consider the circle whose equation is given by

$$(x - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (y - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - (((1 - \sqrt{(x_C-1)^2+y_C^2})/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0.$$

We note that this circle is centered at the incenter of triangle ABC, and we verify that points M, N, and O lie on it since we calculate

$$((1 - \sqrt{(x_C-1)^2+y_C^2})/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - (((1 - \sqrt{(x_C-1)^2+y_C^2})/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$$

$$(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2} - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - (((1 - \sqrt{(x_C-1)^2+y_C^2})/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) = 0,$$

and

$$((1+\sqrt{x_C^2+y_C^2})/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 + (-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C} - \frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 - (((1 - \sqrt{(x_C-1)^2+y_C^2})/2 - \frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2 +$$

$$(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2)=0$$

as desired.

□

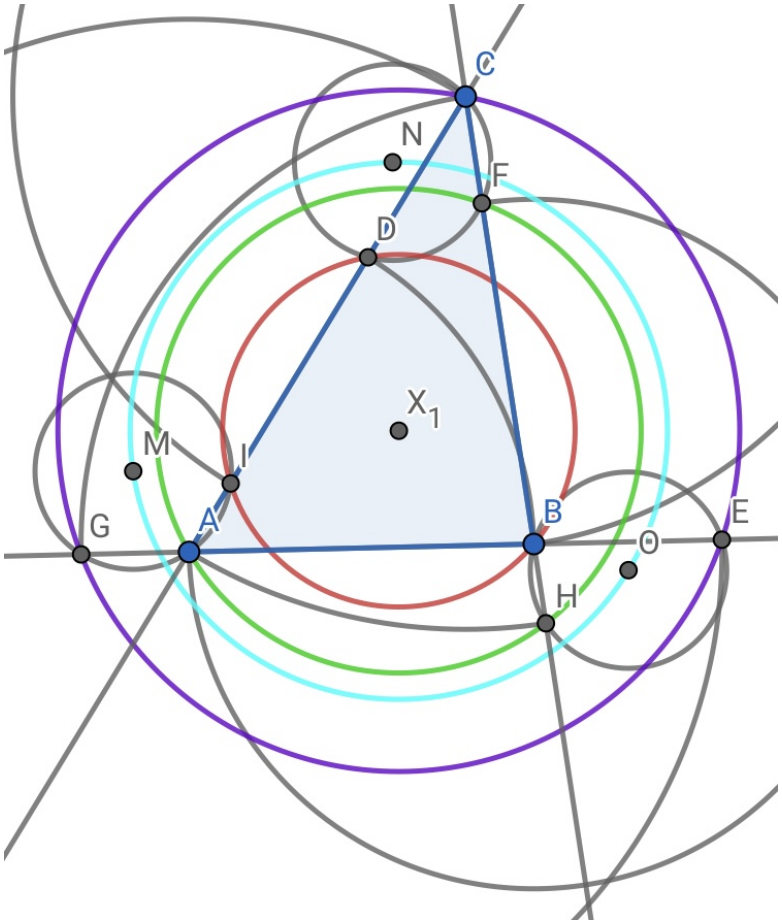

FIGURE 1. Illustration.

REFERENCES

[1] Warren, B.L., "Three Congruent Circles, Six Parallel Lines, X(104), and a New Construction of the Intouch Triangle", *International Journal of Computer Discovered Mathematics*, Under Review.