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Two More Circles in a Bottema-type Configuration

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1. Introduction

In [1], [2], and [3], we constructed a triangle with two similar rectangles erected on the non-base sides and demonstrated the existence of several circles, some of which pass through the Bottema point. Here we will demonstrate the existence of two more circles that can be found in this particular construction.

2. Results

Theorem 2.1. *Let $A, B,$ and C be the vertices of a given triangle.*
Let $ACFG$ form a rectangle, with F adjacent to A.
Let $BCHI$ form a rectangle similar to $ACFG$, with I adjacent to B.
Let lines AG and CF intersect at J.
Let lines BH and CI intersect at K.
Let line AC intersect circle $BCHI$ at L.
Let line BC intersect circle $ACFG$ at M.
Let lines CF and KL intersect at N.
Let lines CI and JM intersect at O.
Then,
1. $F, I, N,$ and O are concyclic and
2. $J, K, N,$ and O are concyclic.

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Proof. A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
F has coordinates $(x_F, -(x_C * x_F)/y_C)$.
G has coordinates $(x_C + x_F, y_C - (x_C * x_F)/y_C)$.
H has coordinates $(x_C - x_F, y_C - ((1 - x_C) * x_F)/y_C)$.
I has coordinates $(1 - x_F, -((1 - x_C) * x_F)/y_C)$.
J has coordinates $((x_C + x_F)/2, (y_C - (x_C * x_F)/y_C)/2)$.
K has coordinates $((x_C - x_F + 1)/2, (y_C - ((1 - x_C) * x_F)/y_C)/2)$.
L has coordinates $((x_C * (x_C - x_F))/(y_C^2 + x_C^2), -((x_F - x_C) * y_C)/(y_C^2 + x_C^2))$.
M has coordinates
$((y_C^2 - (x_C - 1) * x_F)/(y_C^2 + (x_C - 1)^2), -((x_F + x_C - 1) * y_C)/(y_C^2 + (x_C - 1)^2))$.
N has coordinates
$(-(x_F * (x_F - x_C - 1) * y_C^4 + (x_F^3 + (2 * x_C^2 - 4 * x_C - 1) * x_F^2 - x_C * (2 * x_C^2 - x_C - 2) * x_F - x_C^2) * y_C^2 + (x_C - 1) * x_C * x_F^3 + x_C^2 * (x_C^2 - 4 * x_C + 2) * x_F^2 - (x_C - 1)^2 * x_C^3 * x_F)/((y_C^2 + x_C^2) * (y_C^2 + x_F^2)), (x_F * y_C^6 + ((x_C + 2) * x_F^2 + (2 * x_C^2 - 3 * x_C - 1) * x_F + x_C) * y_C^4 + x_C * x_F * (x_F * (x_F + 2 * x_C^2 - 2) + (x_C - 2) * (x_C - 1) * x_C) * y_C^2 + (x_C - 1) * x_C^2 * x_F^3 + (x_C - 1)^2 * x_C^3 * x_F^2)/(y_C * (y_C^2 + x_C^2) * (y_C^2 + x_F^2)))$.
O has coordinates
$(((x_F * (x_F + x_C - 2) + 1) * y_C^4 + x_F * (x_F * (x_F + 2 * x_C^2 - 2) + (x_C - 1) * (2 * x_C^2 - 3 * x_C - 1)) * y_C^2 + (x_C - 1) * x_C * x_F^3 + (x_C - 1)^2 * x_C * (x_C + 2) * x_F^2 + (x_C - 1)^3 * x_C^2 * x_F)/((y_C^2 + (x_C - 1)^2) * (y_C^2 + x_F^2)), (x_F * y_C^6 - (x_F * ((x_C - 3) * x_F - 2 * x_C^2 + x_C + 2) + x_C - 1) * y_C^4 - (x_C - 1) * x_F * (x_F^2 + 2 * x_C^2 * x_F - 4 * x_C * x_F - x_C^3 + x_C) * y_C^2 - (x_C - 1)^2 * x_C * x_F^3 - (x_C - 1)^3 * x_C^2 * x_F^2)/(y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_F^2)))$.
Consider the circle whose equation is given by
$(x - ((x_F * ((2 * x_C - 1) * x_F + 2) - 1) * y_C^4 + 2 * x_C * x_F * ((2 * x_C^2 - 3 * x_C + 2) * x_F + x_C - 1) * y_C^2 + (x_C - 1)^2 * x_C^2 * (2 * x_C - 1) * x_F^2)/(2 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + (y - (x_F * ((2 * x_F - 1) * y_C^4 + 2 * (x_C - 1) * x_C * (2 * x_F - 1) * y_C^2 - x_F^2 + 2 * (x_C - 1) * x_C * (x_C^2 - x_C - 1) * x_F - x_C^4 + 2 * x_C^3 - x_C^2))/(2 * y_C * ((2 * x_F - 1) * y_C^2 + x_F^2 + 2 * (x_C - 1) * x_C * x_F)))^2 - ((x_F - ((x_F * ((2 * x_C - 1) * x_F + 2) - 1) * y_C^4 + 2 * x_C * x_F * ((2 * x_C^2 - 3 * x_C + 2) * x_F + x_C - 1) * y_C^2 + (x_C - 1)^2 * x_C^2 * (2 * x_C - 1) * x_F^2)/(2 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + (-(x_C * x_F)/y_C - (x_F * ((2 * x_F - 1) * y_C^4 + 2 * (x_C - 1) * x_C * (2 * x_F - 1) * y_C^2 - x_F^2 + 2 * (x_C - 1) * x_C * (x_C^2 - x_C - 1) * x_F - x_C^4 + 2 * x_C^3 - x_C^2))/(2 * y_C * ((2 * x_F - 1) * y_C^2 + x_F^2 + 2 * (x_C - 1) * x_C * x_F)))^2) = 0$.
We verify that F, I, N, and O lie on this circle since we calculate
$(x_F - ((x_F * ((2 * x_C - 1) * x_F + 2) - 1) * y_C^4 + 2 * x_C * x_F * ((2 * x_C^2 - 3 * x_C + 2) * x_F + x_C - 1) * y_C^2 + (x_C - 1)^2 * x_C^2 * (2 * x_C - 1) * x_F^2)/(2 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + (-(x_C * x_F)/y_C - (x_F * ((2 * x_F - 1) * y_C^4 + 2 * (x_C - 1) * x_C * (2 * x_F - 1) * y_C^2 - x_F^2 + 2 * (x_C - 1) * x_C * (x_C^2 - x_C - 1) * x_F - x_C^4 + 2 * x_C^3 - x_C^2))/(2 * y_C * ((2 * x_F - 1) * y_C^2 + x_F^2 + 2 * (x_C - 1) * x_C * x_F)))^2 - ((x_F - ((x_F * ((2 * x_C - 1) * x_F + 2) - 1) * y_C^4 + 2 * x_C * x_F * ((2 * x_C^2 - 3 * x_C + 2) * x_F + x_C - 1) * y_C^2 + (x_C - 1)^2 * x_C^2 * (2 * x_C - 1) * x_F^2)/(2 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + (-(x_C * x_F)/y_C - (x_F * ((2 * x_F - 1) * y_C^4 + 2 * (x_C - 1) * x_C * (2 * x_F - 1) * y_C^2 - x_F^2 + 2 * (x_C - 1) * x_C * (x_C^2 - x_C - 1) * x_F - x_C^4 + 2 * x_C^3 - x_C^2))/(2 * y_C * ((2 * x_F - 1) * y_C^2 + x_F^2 + 2 * (x_C - 1) * x_C * x_F)))^2) = 0$,
$(1 - x_F - ((x_F * ((2 * x_C - 1) * x_F + 2) - 1) * y_C^4 + 2 * x_C * x_F * ((2 * x_C^2 - 3 * x_C + 2) * x_F + x_C - 1) * y_C^2 + (x_C - 1)^2 * x_C^2 * (2 * x_C - 1) * x_F^2)/(2 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + (-((1 - x_C) * x_F)/y_C - (x_F * ((2 * x_F - 1) * y_C^4 + 2 * (x_C - 1) * x_C * (2 * x_F - 1) * y_C^2 - x_F^2 + 2 * (x_C - 1) * x_C * (x_C^2 - x_C - 1) * x_F - x_C^4 + 2 * x_C^3 - x_C^2))/(2 * y_C * ((2 * x_F - 1) * y_C^2 + x_F^2 + 2 * (x_C - 1) * x_C * x_F)))^2 - ((x_F - ((x_F * ((2 * x_C - 1) * x_F + 2) -$

$1)*y_C^4+2*x_C*x_F*((2*x_C^2-3*x_C+2)*x_F+x_C-1)*y_C^2+(x_C-1)^2*x_C^2*(2*x_C-1)*x_F^2)/(2*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+(-(x_C*x_F)/y_C-(x_F*((2*x_F-1)*y_C^4+2*(x_C-1)*x_C*(2*x_F-1)*y_C^2-x_F^2+2*(x_C-1)*x_C*(x_C^2-x_C-1)*x_F-x_C^4+2*x_C^3-x_C^2))/(2*y_C*((2*x_F-1)*y_C^2+x_F^2+2*(x_C-1)*x_C*x_F)))^2)=0,$
$(-(x_F*(x_F-x_C-1)*y_C^4+(x_F^3+(2*x_C^2-4*x_C-1)*x_F^2-x_C*(2*x_C^2-x_C-2)*x_F-x_C^2)*y_C^2+(x_C-1)*x_C*x_F^3+x_C^2*(x_C^2-4*x_C+2)*x_F^2-(x_C-1)^2*x_C^3*x_F)/((y_C^2+x_C^2)*(y_C^2+x_F^2))-((x_F*((2*x_C-1)*x_F+2)-1)*y_C^4+2*x_C*x_F*((2*x_C^2-3*x_C+2)*x_F+x_C-1)*y_C^2+(x_C-1)^2*x_C^2*(2*x_C-1)*x_F^2)/(2*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((x_F*y_C^6+((x_C+2)*x_F^2+(2*x_C^2-3*x_C-1)*x_F+x_C)*y_C^4+x_C*x_F*(x_F*(x_F+2*x_C^2-2)+(x_C-2)*(x_C-1)*x_C)*y_C^2+(x_C-1)*x_C^2*x_F^3+(x_C-1)^2*x_C^3*x_F^2)/(y_C*(y_C^2+x_C^2)*(y_C^2+x_F^2))-(x_F*((2*x_F-1)*y_C^4+2*(x_C-1)*x_C*(2*x_F-1)*y_C^2-x_F^2+2*(x_C-1)*x_C*(x_C^2-x_C-1)*x_F-x_C^4+2*x_C^3-x_C^2))/(2*y_C*((2*x_F-1)*y_C^2+x_F^2+2*(x_C-1)*x_C*x_F)))^2-((x_F-((x_F*((2*x_C-1)*x_F+2)-1)*y_C^4+2*x_C*x_F*((2*x_C^2-3*x_C+2)*x_F+x_C-1)*y_C^2+(x_C-1)^2*x_C^2*(2*x_C-1)*x_F^2)/(2*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+(-(x_C*x_F)/y_C-(x_F*((2*x_F-1)*y_C^4+2*(x_C-1)*x_C*(2*x_F-1)*y_C^2-x_F^2+2*(x_C-1)*x_C*(x_C^2-x_C-1)*x_F-x_C^4+2*x_C^3-x_C^2))/(2*y_C*((2*x_F-1)*y_C^2+x_F^2+2*(x_C-1)*x_C*x_F)))^2)=0,$
and
$(((x_F*(x_F+x_C-2)+1)*y_C^4+x_F*(x_F*(x_F+2*x_C^2-2)+(x_C-1)*(2*x_C^2-3*x_C-1))*y_C^2+(x_C-1)*x_C*x_F^3+(x_C-1)^2*x_C*(x_C+2)*x_F^2+(x_C-1)^3*x_C^2*x_F)/((y_C^2+(x_C-1)^2)*(y_C^2+x_F^2))-((x_F*((2*x_C-1)*x_F+2)-1)*y_C^4+2*x_C*x_F*((2*x_C^2-3*x_C+2)*x_F+x_C-1)*y_C^2+(x_C-1)^2*x_C^2*(2*x_C-1)*x_F^2)/(2*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((x_F*y_C^6-(x_F*((x_C-3)*x_F-2*x_C^2+x_C+2)+x_C-1)*y_C^4-(x_C-1)*x_F*(x_F^2+2*x_C^2*x_F-4*x_C*x_F-x_C^3+x_C)*y_C^2-(x_C-1)^2*x_C*x_F^3-(x_C-1)^3*x_C^2*x_F^2)/(y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_F^2))-(x_F*((2*x_F-1)*y_C^4+2*(x_C-1)*x_C*(2*x_F-1)*y_C^2-x_F^2+2*(x_C-1)*x_C*(x_C^2-x_C-1)*x_F-x_C^4+2*x_C^3-x_C^2))/(2*y_C*((2*x_F-1)*y_C^2+x_F^2+2*(x_C-1)*x_C*x_F)))^2-((x_F-((x_F*((2*x_C-1)*x_F+2)-1)*y_C^4+2*x_C*x_F*((2*x_C^2-3*x_C+2)*x_F+x_C-1)*y_C^2+(x_C-1)^2*x_C^2*(2*x_C-1)*x_F^2)/(2*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+(-(x_C*x_F)/y_C-(x_F*((2*x_F-1)*y_C^4+2*(x_C-1)*x_C*(2*x_F-1)*y_C^2-x_F^2+2*(x_C-1)*x_C*(x_C^2-x_C-1)*x_F-x_C^4+2*x_C^3-x_C^2))/(2*y_C*((2*x_F-1)*y_C^2+x_F^2+2*(x_C-1)*x_C*x_F)))^2)=0.$
Next, consider the circle whose equation is given by
$(x-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+(y-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2-(((x_C+x_F)/2-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2)=0.$
We verify that J, K, N, and O lie on this circle since we calculate that
$((x_C+x_F)/2-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2-(((x_C+x_F)/2-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2)=0,$
$((x_C-x_F+1)/2-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((y_C-((1-x_C)*x_F)/y_C)/2-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2-(((x_C+x_F)/2-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2)=0,$
$(-(x_F*(x_F-x_C-1)*y_C^4+(x_F^3+(2*x_C^2-4*x_C-1)*x_F^2-x_C*(2*x_C^2-x_C-$

$2)*x_F-x_C^2)*y_C^2+(x_C-1)*x_C*x_F^3+x_C^2*(x_C^2-4*x_C+2)*x_F^2-(x_C-1)^2*x_C^3*x_F)/((y_C^2+x_C^2)*(y_C^2+x_F^2))-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((x_F*y_C^6+((x_C+2)*x_F^2+(2*x_C^2-3*x_C-1)*x_F+x_C)*y_C^4+x_C*x_F*(x_F*(x_F+2*x_C^2-2)+(x_C-2)*(x_C-1)*x_C)*y_C^2+(x_C-1)*x_C^2*x_F^3+(x_C-1)^2*x_C^3*x_F^2)/(y_C*(y_C^2+x_C^2)*(y_C^2+x_F^2))-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2-(((x_C+x_F)/2-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2)=0,$
$(((x_F*(x_F+x_C-2)+1)*y_C^4+x_F*(x_F*(x_F+2*x_C^2-2)+(x_C-1)*(2*x_C^2-3*x_C-1))*y_C^2+(x_C-1)*x_C*x_F^3+(x_C-1)^2*x_C*(x_C+2)*x_F^2+(x_C-1)^3*x_C^2*x_F)/((y_C^2+(x_C-1)^2)*(y_C^2+x_F^2))-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((x_F*y_C^6-(x_F*((x_C-3)*x_F-2*x_C^2+x_C+2)+x_C-1)*y_C^4-(x_C-1)*x_F*(x_F^2+2*x_C^2*x_F-4*x_C*x_F-x_C^3+x_C)*y_C^2-(x_C-1)^2*x_C*x_F^3-(x_C-1)^3*x_C^2*x_F^2)/(y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_F^2))-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2-(((x_C+x_F)/2-(((x_C-1)*x_C*(2*x_C-1)*x_F)/(4*y_C^2)+((2*x_C-1)*x_F+2*x_C+1)/4))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F+1)*y_C^2+(2*x_C^2-2*x_C-1)*x_F-(x_C-1)*x_C)/(4*y_C))^2)=0$
as desired. □

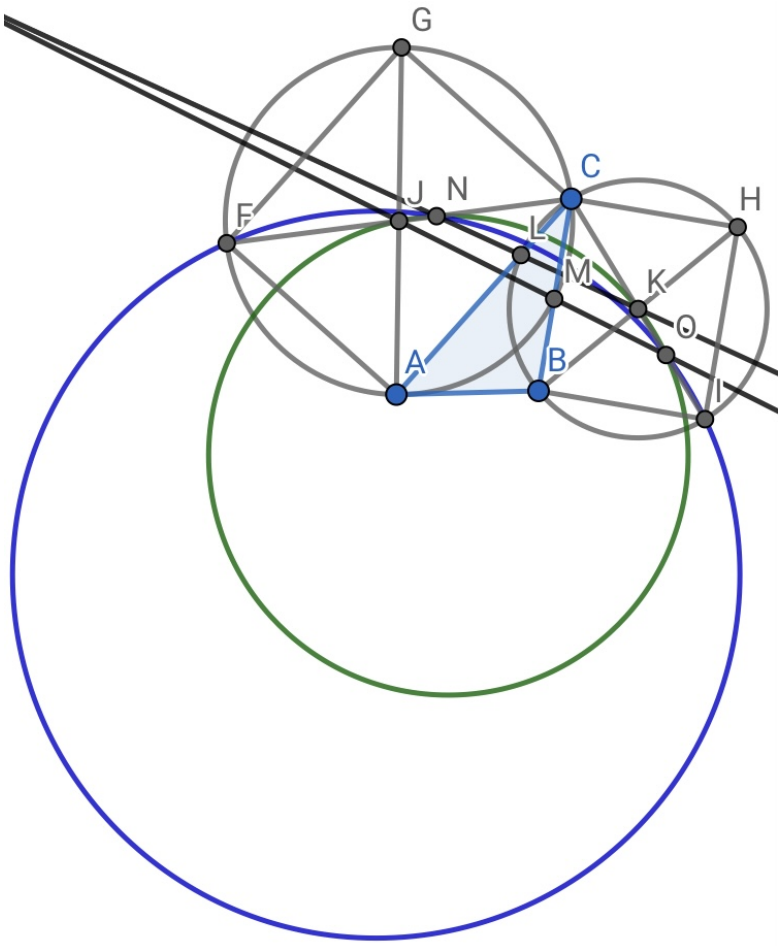

FIGURE 1. Illustration.

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