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The Diagonal Intersection Lies on the Radical Line of Two Certain Circles in a Quadrilateral

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1. Introduction

In [1], some results were discussed regarding a phenomenon where the diagonal intersection in a given quadrilateral lies on the radical line of two circles: the circles passing through opposite pairs of vertex midpoints and the points that complete the quadrilateral. Here, we prove a generalization of this.

2. Preliminaries

Let A, B, C, and D be the vertices of a given quadrilateral arranged counterclockwise.
Let line AD intersect line BC at point E.
Let line AB intersect line CD at point F.
Let G be the intersection point of the diagonals.
Let α be any given point on line AD.
Let β be the point on line BC such that $C\beta : B\beta = A\alpha : D\alpha$.
Let γ be the point on line AB such that $A\gamma : B\gamma = A\alpha : D\alpha$.
Let δ be the point on line CD such that $C\delta : D\delta = A\alpha : D\alpha$.
Let $\mathcal{P}$ denote the circle passing through α, β, and E.
Let $\mathcal{Q}$ denote the circle passing through γ, δ, and F.

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3. Results

Theorem 3.1. *G lies on the radical line of $\mathcal{P}$ and $\mathcal{Q}$.*

Proof. Without loss of generality, we can orient the quadrilateral on a Cartesian grid to satisfy the following.
Let A have coordinates given by $(0,0)$.
Let B have coordinates given by $(1,0)$.
Let C have coordinates given by (x_C, y_C).
Let D have coordinates given by (x_D, y_D).
Line AD is given by the equation $y = x(\frac{y_D}{x_D})$.
Line BC is given by the equation $y = (\frac{y_C}{x_C - 1})(x-1)$.
Line AB is given by the equation $y = 0$.
Line CD is given by the equation $y = y_C + (\frac{y_D - y_C}{x_D - x_C})(x - x_C)$.
Line AC is given by the equation $y = x(\frac{y_C}{x_C})$.
Line BD is given by the equation $y = (\frac{y_D}{x_D - 1})(x-1)$.
E has coordinates given by $x_E = \frac{x_D y_C}{x_D y_C - x_C y_D + y_D}$ and $y_E = \frac{y_C y_D}{x_D y_C - x_C y_D + y_D}$.
F has coordinates given by $x_F = \frac{x_D y_C - x_C y_D}{y_C - y_D}$ and $y_F = 0$.
G has coordinates given by $x_G = \frac{x_C y_D}{x_C y_D - x_D y_C + y_C}$ and $y_G = \frac{y_C y_D}{x_C y_D - x_D y_C + y_C}$.
α has coordinates given by $(x_\alpha, x_\alpha(\frac{y_D}{x_D}))$.
β has coordinates given by $(x_C - (x_C - 1)(\frac{x_\alpha}{x_D}), y_C(1 - \frac{x_\alpha}{x_D}))$.
γ has coordinates given by $(\frac{x_\alpha}{x_D}, 0)$.
δ has coordinates given by $(x_C - (x_C - x_D)(\frac{x_\alpha}{x_D}), y_C + (y_D - y_C)(\frac{x_\alpha}{x_D}))$.
$\mathcal{P}$ is given by the equation
$(x + \frac{x_\alpha y_C y_D^2 + y_D((x_\alpha - x_D)y_C^2 + (x_C-1)^2 x_\alpha - (x_C-1)x_C x_D) + x_D^2 y_C(x_\alpha + 1)}{2x_D(y_D(x_C-1) - x_D y_C)})^2$
$+ (y - \frac{x_\alpha y_D^2(x_C-1) - x_D y_C y_D + x_D y_C^2(x_\alpha - x_D) + x_\alpha x_D(x_C-1)(x_C + x_D - 1) - x_C x_D^2(x_C-1)}{2x_D(y_D(x_C-1) - x_D y_C)})^2$
$= (x_\alpha + \frac{x_\alpha y_C y_D^2 + y_D((x_\alpha - x_D)y_C^2 + (x_C-1)^2 x_\alpha - (x_C-1)x_C x_D) + x_D^2 y_C(x_\alpha + 1)}{2x_D(y_D(x_C-1) - x_D y_C)})^2$
$+ (x_\alpha(\frac{y_D}{x_D}) - \frac{x_\alpha y_D^2(x_C-1) - x_D y_C y_D + x_D y_C^2(x_\alpha - x_D) + x_\alpha x_D(x_C-1)(x_C + x_D - 1) - x_C x_D^2(x_C-1)}{2x_D(y_D(x_C-1) - x_D y_C)})^2$.
$\mathcal{Q}$ is given by the equation
$(x - \frac{1}{2}(\frac{x_D y_C - x_C y_D}{y_C - y_D} + \frac{x_a}{x_D}))^2$
$+ (y - \frac{x_\alpha y_D^2 - (2x_\alpha - x_D)y_C y_D + (x_\alpha - x_D)y_C^2 + (x_D - x_C - 1)(x_D - x_C)x_\alpha + x_C x_D^2 - x_C^2 x_D}{2x_D(y_D - y_C)})^2$
$= (\frac{x_\alpha}{x_D} - \frac{1}{2}(\frac{x_D y_C - x_C y_D}{y_C - y_D} + \frac{x_a}{x_D}))^2$
$+ (\frac{x_\alpha y_D^2 - (2x_\alpha - x_D)y_C y_D + (x_\alpha - x_D)y_C^2 + (x_D - x_C - 1)(x_D - x_C)x_\alpha + x_C x_D^2 - x_C^2 x_D}{2x_D(y_D - y_C)})^2$.
It follows that the radical line between $\mathcal{P}$ and $\mathcal{Q}$ is given by the equation
$(x_\alpha y_C y_D^3 - (x_D y_C^2 - (x_C - 1)x_C x_\alpha)y_D^2 - y_C(x_\alpha y_C^2 - x_D y_C^2 - x_D^2 x_\alpha + x_D x_\alpha + x_C^2 x_\alpha - x_C x_\alpha + 2x_C x_D^2 - 2x_D^2 - x_C^2 x_D + x_C x_D)y_D - (x_D - 1)x_D(x_\alpha - x_D)y_C^2)x - x_\alpha y_C y_D^3 + (x_\alpha y_C^2 - ((x_D + x_C - 1)x_\alpha - x_C x_D)y y_C - (x_C - 1)x_C x_\alpha)y_D^2 + (((x_D + x_C - 1)x_\alpha - x_C x_D)y y_C^2 - x_D(x_D - 2x_C + 1)x_\alpha y_C - (x_C - 1)x_C((3x_D - x_C - 1)x_\alpha - x_D(2x_D - x_C))y)y_D - (x_D - 1)x_D((x_D - 3x_C + 1)x_\alpha + x_C x_D)y y_C = 0$.
We then verify that G indeed lies on this line since we calculate
$(x_\alpha y_C y_D^3 - (x_D y_C^2 - (x_C - 1)x_C x_\alpha)y_D^2 - y_C(x_\alpha y_C^2 - x_D y_C^2 - x_D^2 x_\alpha + x_D x_\alpha + x_C^2 x_\alpha - x_C x_\alpha + 2x_C x_D^2 - 2x_D^2 - x_C^2 x_D + x_C x_D)y_D - (x_D - 1)x_D(x_\alpha - x_D)y_C^2)(\frac{x_C y_D}{x_C y_D - x_D y_C + y_C}) - x_\alpha y_C y_D^3 + (x_\alpha y_C^2 - ((x_D + x_C - 1)x_\alpha - x_C x_D)(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C})y_C - (x_C - 1)x_C x_\alpha)y_D^2 + (((x_D + x_C - 1)x_\alpha - x_C x_D)(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C})y_C^2 - x_D(x_D - 2x_C + 1)x_\alpha y_C - (x_C -$

$1)x_C((3x_D - x_C - 1)x_\alpha - x_D(2x_D - x_C))(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C}))y_D - (x_D - 1)x_D((x_D - 3x_C + 1)x_\alpha + x_C x_D)(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C})y_C = 0$ as desired. □

Theorem 3.2. *If the quadrilateral is cyclic, then the power center of P, Q, and the circumcircle is G.*

Proof. Suppose that A, B, C, and D are cyclic.
The circumcircle is given by the equation $(x - \frac{1}{2})^2 + y^2 + \frac{y(x_C - x_C^2 - y_C^2)}{y_C} = \frac{1}{4}$.
This indicates that x_D and y_D must satisfy $(x_D - \frac{1}{2})^2 + y_D^2 + \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C} = \frac{1}{4}$.
The radical line between $\mathcal{Q}$ and the circumcircle is given by the equation $((x_a + (x_C - 1)x_D)y_C y_D + (-x_a - x_D^2 + x_D)y_C^2)x + x_a y y_C y_D^2 + (-2x_a y y_C^2 - x_C x_a y_C + (x_C - x_C^2)x_D y)y_D + x_a y y_C^3 + x_D x_a y_C^2 + ((x_D^2 + (-2x_C - 1)x_D + x_C^2 + x_C)x_a + x_C x_D^2 - x_C x_D)y y_C = 0$.
Using the fact that $x_D = \frac{1}{2} \pm \sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}}$ we then verify that G indeed lies on this line since we calculate

$$((x_a+(x_C-1)x_D)y_C y_D+(-x_a-x_D^2+x_D)y_C^2)(\frac{x_C y_D}{x_C y_D - x_D y_C + y_C})+x_a(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C})y_C y_D^2+$$
$$(-2x_a(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C})y_C^2 - x_C x_a y_C + (x_C - x_C^2)x_D(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C}))y_D + x_a(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C})y_C^3 +$$
$$x_D x_a y_C^2 + ((x_D^2 + (-2x_C - 1)x_D + x_C^2 + x_C)x_a + x_C x_D^2 - x_C x_D)(\frac{y_C y_D}{x_C y_D - x_D y_C + y_C})y_C$$
$$= ((x_a+(x_C-1)(\tfrac{1}{2}+\sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}}))y_C y_D+(-x_a-(\tfrac{1}{2}+\sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})^2$$
$$+ (\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}}))y_C^2)(\frac{x_C y_D}{x_C y_D - (\frac{1}{2}+\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})$$
$$+ x_a(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}+\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})y_C y_D^2$$
$$+ (-2x_a(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}+\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})y_C^2 - x_C x_a y_C$$
$$+(x_C - x_C^2)(\tfrac{1}{2}+\sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}+\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C}))y_D +$$
$$x_a(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}+\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})y_C^3 + (\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})x_a y_C^2 +$$
$$(((\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})^2 + (-2x_C - 1)(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}}) +$$
$$x_C^2 + x_C)x_a + x_C(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})^2$$
$$- x_C(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}}))(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}+\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})y_C = 0$$

and

$$((x_a+(x_C-1)(\tfrac{1}{2}-\sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}}))y_C y_D+(-x_a-(\tfrac{1}{2}-\sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})^2$$
$$+ (\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}}))y_C^2)(\frac{x_C y_D}{x_C y_D - (\frac{1}{2}-\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})$$
$$+ x_a(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}-\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})y_C y_D^2$$
$$+ (-2x_a(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}-\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C})y_C^2 - x_C x_a y_C$$
$$+(x_C - x_C^2)(\tfrac{1}{2}-\sqrt{\tfrac{1}{4} - y_D^2 - \tfrac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})(\frac{y_C y_D}{x_C y_D - (\frac{1}{2}-\sqrt{\frac{1}{4} - y_D^2 - \frac{y_D(x_C - x_C^2 - y_C^2)}{y_C}})y_C + y_C}))y_D +$$

$$x_a\left(\frac{y_C y_D}{x_C y_D-\left(\frac{1}{2}-\sqrt{\frac{1}{4}-y_D^2-\frac{y_D(x_C-x_C^2-y_C^2)}{y_C}}\right)y_C+y_C}\right)y_C^3+\left(\frac{1}{2}-\sqrt{\frac{1}{4}-y_D^2-\frac{y_D(x_C-x_C^2-y_C^2)}{y_C}}\right)x_a y_C^2+$$
$$\left(\left(\left(\frac{1}{2}-\sqrt{\frac{1}{4}-y_D^2-\frac{y_D(x_C-x_C^2-y_C^2)}{y_C}}\right)^2+(-2x_C-1)\left(\frac{1}{2}-\sqrt{\frac{1}{4}-y_D^2-\frac{y_D(x_C-x_C^2-y_C^2)}{y_C}}\right)+\right.\right.$$
$$\left.\left.x_C^2+x_C\right)x_a+x_C\left(\frac{1}{2}-\sqrt{\frac{1}{4}-y_D^2-\frac{y_D(x_C-x_C^2-y_C^2)}{y_C}}\right)^2\right.$$
$$\left.-x_C\left(\frac{1}{2}-\sqrt{\frac{1}{4}-y_D^2-\frac{y_D(x_C-x_C^2-y_C^2)}{y_C}}\right)\right)\left(\frac{y_C y_D}{x_C y_D-\left(\frac{1}{2}-\sqrt{\frac{1}{4}-y_D^2-\frac{y_D(x_C-x_C^2-y_C^2)}{y_C}}\right)y_C+y_C}\right)y_C=0.$$

Combining this with the previous result and the fact that the radical lines of any three circles in the plane are concurrent gives the desired result. □

The results in [1] are the special cases of these results where we take α to be the midpoint of A and D.

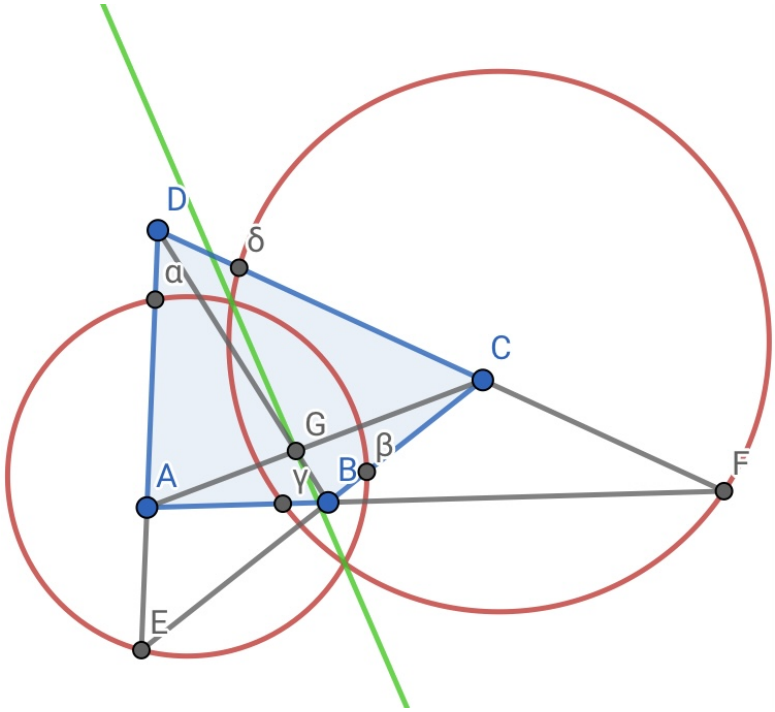

Figure 1. Given that $A\alpha : D\alpha = C\beta : B\beta = A\gamma : B\gamma = C\delta : D\delta$, then the intersection point of the diagonals lies on the radical line of the circle passing through α, β, and E and the circle passing through γ, δ, and F.

References

[1] Warren, B.L., "The Power Center of Three Circles in a Cyclic Quadrilateral is the Diagonal Intersection Point", *International Journal of Computer Discovered Mathematics*, Vol 8, paper 20, pp. 108-112.