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Three Congruent Circles, Six Parallel Lines, X(104), and a New Construction of the Intouch Triangle

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1. Introduction

Take an edge of a triangle and trace out that length on the other two sides. If this is done in a certain way, for all the sides, then the existence of a number of deducible figures can be demonstrated in the construction.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be the point on line AC, with $\angle CAB = \angle DAB$, such that $AD = AB$.
Let E be the point on line AB, with $\angle CAB = \angle CAE$, such that $AE = AC$.
Let F be the point on line BC, with $\angle CBA = \angle FBA$, such that $BF = AB$.
Let G be the point on line AB, with $\angle CBA = \angle CBG$, such that $BG = BC$.
Let H be the point on line BC, with $\angle ACB = \angle ACH$, such that $CH = AC$.
Let I be the point on line AC, with $\angle ACB = \angle ICB$, such that $CI = BC$.
Let J be the midpoint of E and G.
Let K be the midpoint of D and I.
Let L be the midpoint of F and H.
Let M be the circumcenter of A, G, and I.
Let N be the circumcenter of C, D, and F.
Let O be the circumcenter of B, E, and H.

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Let Q be the midpoint of D and F.
Let R be the midpoint of G and I.
Let S be the midpoint of E and H.
Let circles ABC and AGI intersect at T.
Let circles ABC and CDF intersect at U.
Let circles ABC and BEH intersect at V.
Then,
1. J, K, and L form the intouch triangle,
2. Lines DF, EH, and GI are parallel,
3. Circles AGI, CDF, and BEH are congruent,
4. Lines AM, BO, and CN are concurrent at $X(104)$ which lies on the circumcircle of ABC,
5. M, R, and T are collinear, N, Q, and U are collinear, and O, S, and V are collinear, and
6. Lines MRT, NQU, and OSV are parallel.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates $(\frac{x_C}{\sqrt{x_C^2+y_C^2}}, \frac{y_C}{\sqrt{x_C^2+y_C^2}})$.
E has coordinates $(\sqrt{x_C^2+y_C^2}, 0)$.
F has coordinates $(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1, \frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})$.
G has coordinates $(1-\sqrt{(x_C-1)^2+y_C^2}, 0)$.
H has coordinates $(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}, y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))$.
I has coordinates $(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}), y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))$.
J has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2}+\sqrt{x_C^2+y_C^2})/2, 0)$.
K has coordinates $(\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}, \frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})$.
L has coordinates $((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2, (\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)$.
M has coordinates $((1-\sqrt{(x_C-1)^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
N has coordinates $(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}, -\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})$.
O has coordinates $((1+\sqrt{x_C^2+y_C^2})/2, -\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.
Q has coordinates $((\frac{x_C}{\sqrt{x_C^2+y_C^2}}+\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1)/2, (\frac{y_C}{\sqrt{x_C^2+y_C^2}}+\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})/2)$.
R has coordinates
$((1-\sqrt{(x_C-1)^2+y_C^2}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2, (y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)$.
S has coordinates $((\sqrt{x_C^2+y_C^2}+x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2, (y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)$.

1. We verify that J, K, and L respectively lie on lines AB, AC, and BC since we calculate that
$(\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}})*(\frac{y_C}{x_C})-\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}=0$,

$y_J = 0$, and
$((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2-1)*(\frac{y_C}{x_C-1})-((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2) =$
$0.$
The incircle of A, B, and C is given by the equation
$(x-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(y-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2) =$
$0.$
We then verify that J, K, and L lie on the incircle since we calculate that
$((1-\sqrt{(x_C-1)^2+y_C^2}+\sqrt{x_C^2+y_C^2})/2-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2$
$+(0-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2)=0,$
$(\frac{x_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+(\frac{y_C(1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2})}{2*\sqrt{x_C^2+y_C^2}}-$
$\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2)=0,$
and
$((1+x_C+\frac{x_C-1+(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2-\frac{x_C+\sqrt{x_C^2+y_C^2}}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2+((\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}+$
$y_C(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2-\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2-((\frac{y_C}{1+\sqrt{x_C^2+y_C^2}+\sqrt{(x_C-1)^2+y_C^2}})^2)=$
$0.$

2. The slope of line DF is $\frac{y_D-y_F}{x_D-x_F}=\frac{\frac{y_C}{\sqrt{x_C^2+y_C^2}}-(\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})}{\frac{x_C}{\sqrt{x_C^2+y_C^2}}-(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1)}.$
The slope of line EH is $\frac{y_E-y_H}{x_E-x_H}=\frac{-(y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))}{\sqrt{x_C^2+y_C^2}-(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})}.$
The slope of line GI is $\frac{y_G-y_I}{x_G-x_I}=\frac{-(y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))}{1-\sqrt{(x_C-1)^2+y_C^2}-(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))}.$
We then verify that all these slopes are the same since we calculate that
$\frac{\frac{y_C}{\sqrt{x_C^2+y_C^2}}-(\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})}{\frac{x_C}{\sqrt{x_C^2+y_C^2}}-(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1)}-(\frac{-(y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))}{\sqrt{x_C^2+y_C^2}-(x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})})=0$
and
$\frac{\frac{y_C}{\sqrt{x_C^2+y_C^2}}-(\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})}{\frac{x_C}{\sqrt{x_C^2+y_C^2}}-(\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1)}-(\frac{-(y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))}{1-\sqrt{(x_C-1)^2+y_C^2}-(x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))})=0.$

3. Circle ABC is given by the equation
$(x-\frac{1}{2})^2+(y-\frac{x_C(x_C-1)+y_C^2}{2y_C})^2-(\frac{1}{4}+(\frac{x_C(x_C-1)+y_C^2}{2y_C})^2)=0.$
Circle AGI is given by the equation
$(x-((1-\sqrt{(x_C-1)^2+y_C^2})/2))^2+(y-(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))^2-$
$((1-\sqrt{(x_C-1)^2+y_C^2})/2)^2-(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})^2=0.$
Circle CDF is given by the equation

$(x-(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}))^2+(y-(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}))^2-$
$((x_C-(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}))^2+(y_C-(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}))^2) =$
$0.$

Circle BEH is given by the equation

$(x-((1+\sqrt{x_C^2+y_C^2})/2))^2+(y-(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}))^2-$
$(1-((1+\sqrt{x_C^2+y_C^2})/2))^2-(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})^2=0.$

T has coordinates $(\frac{1-\sqrt{x_C^2+y_C^2}}{2}, \frac{y_C^2+(x_C-1)*(x_C+\sqrt{x_C^2+y_C^2})}{2y_C})$.

U has coordinates $(1/2, \frac{x_C*(x_C-1)+y_C^2+(\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.

V has coordinates $(\frac{1+\sqrt{(x_C-1)^2+y_C^2}}{2}, y_C/2-\frac{x_C*(1-x_C+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})$.

We verify that circles AGI and CDF are congruent since we calculate that

$(x_N-x_C)^2+(y_N-y_C)^2-x_M^2-y_M^2$
$=((\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2})-x_C)^2+(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}-$
$y_C)^2-((1-\sqrt{(x_C-1)^2+y_C^2})/2)^2-(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})^2=$
0 and we calculate that circles AGI and BEH are congruent since we have that

$(x_O-1)^2+y_O^2-x_M^2-y_M^2$
$=(((1+\sqrt{x_C^2+y_C^2})/2)-1)^2+(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})^2-((1-$
$\sqrt{(x_C-1)^2+y_C^2})/2)^2-(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})^2=0.$

4. Line AM is given by the equation

$y=x*(\frac{y_M}{x_M})=x*(\frac{-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2})$.

Line BO is given by the equation

$y=(x-1)*(\frac{y_O}{x_O-1})=(x-1)*(\frac{-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1})$.

Line CN is given by the equation

$y=y_C+(x-x_C)*(\frac{y_N-y_C}{x_N-x_C})=y_C+(x-x_C)*(\frac{-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}-y_C}{\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}-x_C})$.

Consider the point P which is $X(104)$ whose coordinates are given by

$$\left(\frac{\left(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}\right)}{\left(\frac{-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)-\left(\left(\frac{-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)\right)},\right.$$
$$\left.\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)*\left(\frac{-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\frac{-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}-\left(\frac{-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)}\right).$$

We verify that P lies on lines AM, BO, and CN since we respectively calculate that

$$\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)-\left(\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)\right)}*$$
$$\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)$$
$$-\left(\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)*\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}-\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)}\right)=$$
$$0,$$
$$\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)-\left(\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)\right)}-$$
$$1)*\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)$$
$$-\left(\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)*\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}-\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)}\right)=$$
$$0,$$
and
$$y_C+\left(\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)-\left(\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)\right)}-\right.$$
$$\left.x_C\right)*\left(\frac{-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)-y_C^2}{2y_C}-y_C}{\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}-x_C}\right)$$
$$-\left(\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)*\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}-\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)}\right)=$$
$$0.$$
We also verify that P lies on the circumcircle of ABC since we calculate that
$$\frac{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)}{\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(\sqrt{x_C^2+y_C^2}\right)*\left(x_C-1+\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}\right)-\left(\left(\frac{-\frac{x_C-x_C^2-y_C^2+\left(-x_C+\sqrt{x_C^2+y_C^2}\right)*\left(\sqrt{(x_C-1)^2+y_C^2}\right)}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}\right)\right)}-$$

$$\frac{1}{2})^2+(\frac{(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})*(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})}{(1-\sqrt{(x_C-1)^2+y_C^2})/2}$$
$$\frac{-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1+\sqrt{x_C^2+y_C^2})/2-1}-(\frac{-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}}{(1-\sqrt{(x_C-1)^2+y_C^2})/2})$$
$$\frac{x_C(x_C-1)+y_C^2}{2y_C})^2-(\frac{1}{4}+(\frac{x_C(x_C-1)+y_C^2}{2y_C})^2)=0.$$

5. Line MR is given by the equation

$$y=y_M+(x-x_M)*(\frac{y_M-y_R}{x_M-x_R})=(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})+(x-$$
$$((1-\sqrt{(x_C-1)^2+y_C^2})/2))*(\frac{(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})-((y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)}{((1-\sqrt{(x_C-1)^2+y_C^2})/2)-((1-\sqrt{(x_C-1)^2+y_C^2}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)}).$$

We verify that T lies on line MR since we calculate that

$$(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})+(\frac{1-\sqrt{x_C^2+y_C^2}}{2}-((1-\sqrt{(x_C-1)^2+y_C^2})/2))*$$
$$(\frac{(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})-((y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)}{((1-\sqrt{(x_C-1)^2+y_C^2})/2)-((1-\sqrt{(x_C-1)^2+y_C^2}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)})-(\frac{y_C^2+(x_C-1)*(x_C+\sqrt{x_C^2+y_C^2})}{2y_C})=$$
$0.$

Line NQ is given by the equation

$$y=y_N+(x-x_N)*(\frac{y_N-y_Q}{x_N-x_Q})=(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})+(x-$$
$$(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}))*(\frac{(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})-((\frac{y_C}{\sqrt{x_C^2+y_C^2}}+\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})/2)}{(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2})-((\frac{x_C}{\sqrt{x_C^2+y_C^2}}+\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1)/2)}).$$

We verify that U lies on line NQ since we calculate that

$$(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})+(1/2-(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}))*$$
$$(\frac{(-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C})-((\frac{y_C}{\sqrt{x_C^2+y_C^2}}+\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}})/2)}{(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2})-((\frac{x_C}{\sqrt{x_C^2+y_C^2}}+\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1)/2)})$$
$$-(\frac{x_C*(x_C-1)+y_C^2+(\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C})=0.$$

Line OS is given by the equation

$$y=y_O+(x-x_O)*(\frac{y_O-y_S}{x_O-x_S})=(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})+(x-((1+$$
$$\sqrt{x_C^2+y_C^2})/2))*(\frac{(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})-((y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)}{((1+\sqrt{x_C^2+y_C^2})/2)-((\sqrt{x_C^2+y_C^2}+x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2)}).$$

We verify that V lies on line OS since we calculate that

$$(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})+(\frac{1+\sqrt{(x_C-1)^2+y_C^2}}{2}-((1+\sqrt{x_C^2+y_C^2})/2))*$$
$$(\frac{(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C})-((y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2)}{((1+\sqrt{x_C^2+y_C^2})/2)-((\sqrt{x_C^2+y_C^2}+x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2)})$$

$$-\left(y_C/2-\frac{x_C*(1-x_C+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}\right)=0.$$

6. The slope of line MRT is $\frac{y_M-y_R}{x_M-x_R}$

$$=\frac{\left(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}\right)-\left((y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}})\right)/2)}{((1-\sqrt{(x_C-1)^2+y_C^2})/2)-((1-\sqrt{(x_C-1)^2+y_C^2}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)}.$$

The slope of line NQU is $\frac{y_N-y_Q}{x_N-x_Q}$

$$=\frac{-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}-\left(\left(\frac{y_C}{\sqrt{x_C^2+y_C^2}}+\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}\right)/2\right)}{\left(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}\right)-\left(\left(\frac{x_C}{\sqrt{x_C^2+y_C^2}}+\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1\right)/2\right)}.$$

The slope of line OSV is $\frac{y_O-y_S}{x_O-x_S}$

$$=\frac{\left(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}\right)-\left((y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2\right)}{((1+\sqrt{x_C^2+y_C^2})/2)-((\sqrt{x_C^2+y_C^2}+x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2)}.$$

We then verify that all these slopes are the same since we calculate that

$$\frac{\left(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}\right)-\left((y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2\right)}{((1-\sqrt{(x_C-1)^2+y_C^2})/2)-((1-\sqrt{(x_C-1)^2+y_C^2}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)}$$

$$-\left(\frac{-\frac{(x_C-1)*\sqrt{x_C^2+y_C^2}-x_C*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})-y_C^2}{2y_C}-\left(\left(\frac{y_C}{\sqrt{x_C^2+y_C^2}}+\frac{y_C}{\sqrt{(x_C-1)^2+y_C^2}}\right)/2\right)}{\left(\frac{1+\sqrt{x_C^2+y_C^2}-\sqrt{(x_C-1)^2+y_C^2}}{2}\right)-\left(\left(\frac{x_C}{\sqrt{x_C^2+y_C^2}}+\frac{x_C-1}{\sqrt{(x_C-1)^2+y_C^2}}+1\right)/2\right)}\right)=0$$

and

$$\frac{\left(-\frac{x_C-x_C^2-y_C^2+(-x_C+\sqrt{x_C^2+y_C^2})*(\sqrt{(x_C-1)^2+y_C^2})}{2y_C}\right)-\left((y_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2\right)}{((1-\sqrt{(x_C-1)^2+y_C^2})/2)-((1-\sqrt{(x_C-1)^2+y_C^2}+x_C*(1-\frac{\sqrt{(x_C-1)^2+y_C^2}}{\sqrt{x_C^2+y_C^2}}))/2)}$$

$$-\left(\frac{\left(-\frac{x_C-x_C^2-y_C^2+(\sqrt{x_C^2+y_C^2})*(x_C-1+\sqrt{(x_C-1)^2+y_C^2})}{2y_C}\right)-\left((y_C*(1-\frac{\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}}))/2\right)}{((1+\sqrt{x_C^2+y_C^2})/2)-((\sqrt{x_C^2+y_C^2}+x_C+\frac{(1-x_C)*\sqrt{x_C^2+y_C^2}}{\sqrt{(x_C-1)^2+y_C^2}})/2)}\right)=0.$$

□

References

[1] Weisstein, Eric W. "Kimberling Center." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/KimberlingCenter.html

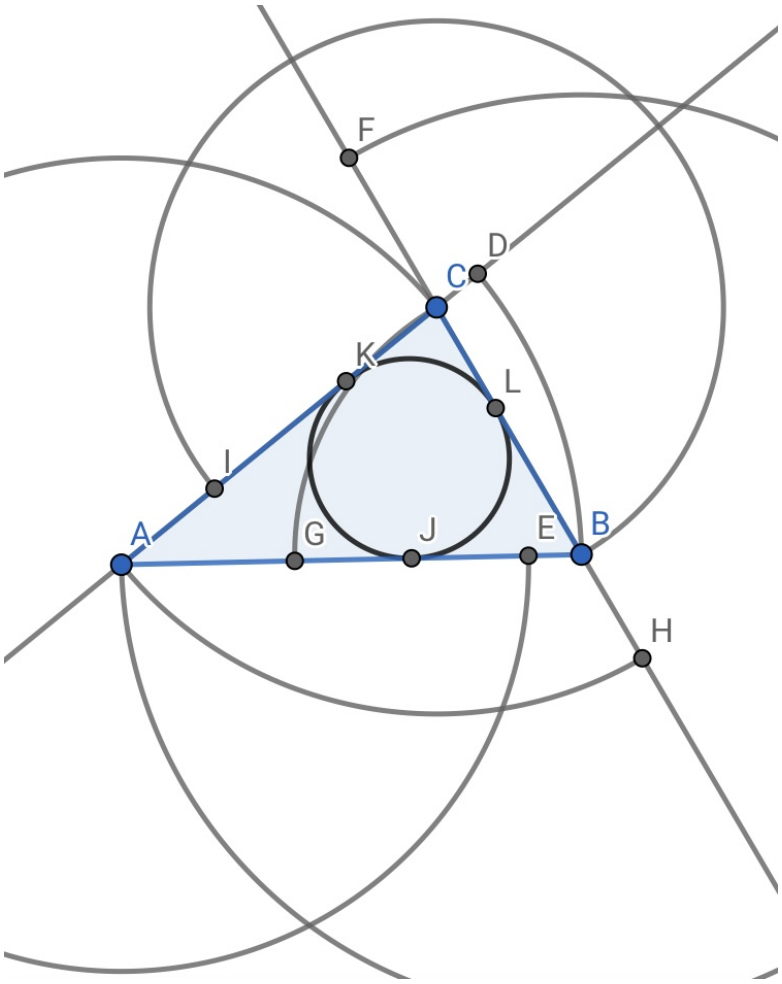

FIGURE 1. J, K, and L form the intouch triangle.

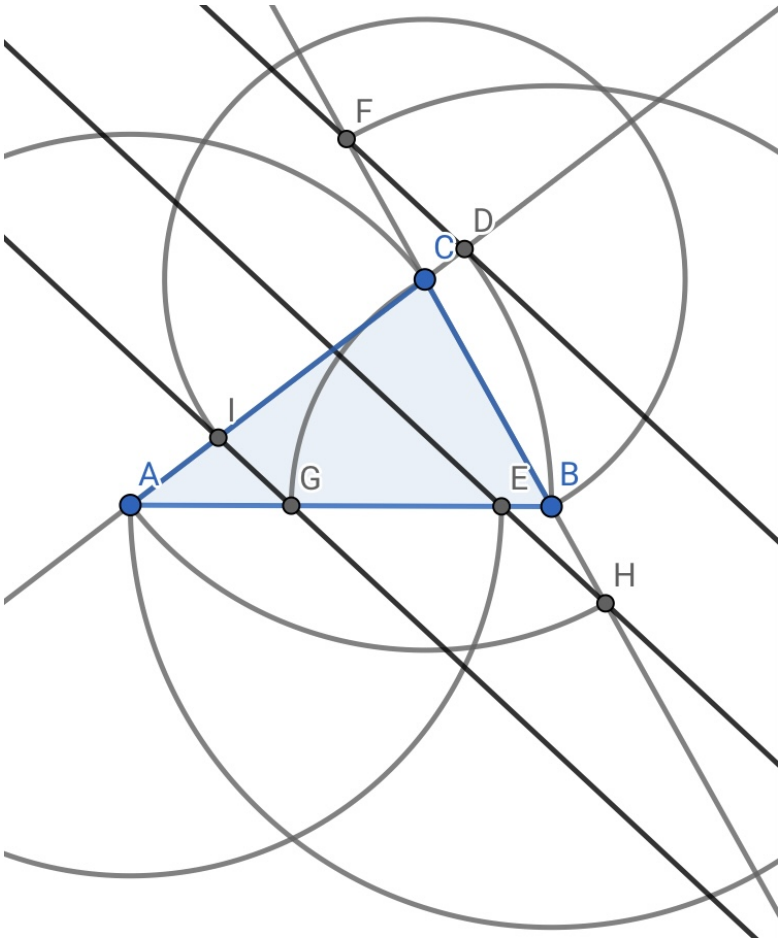

Figure 2. Lines DF, EH, and GI are parallel.

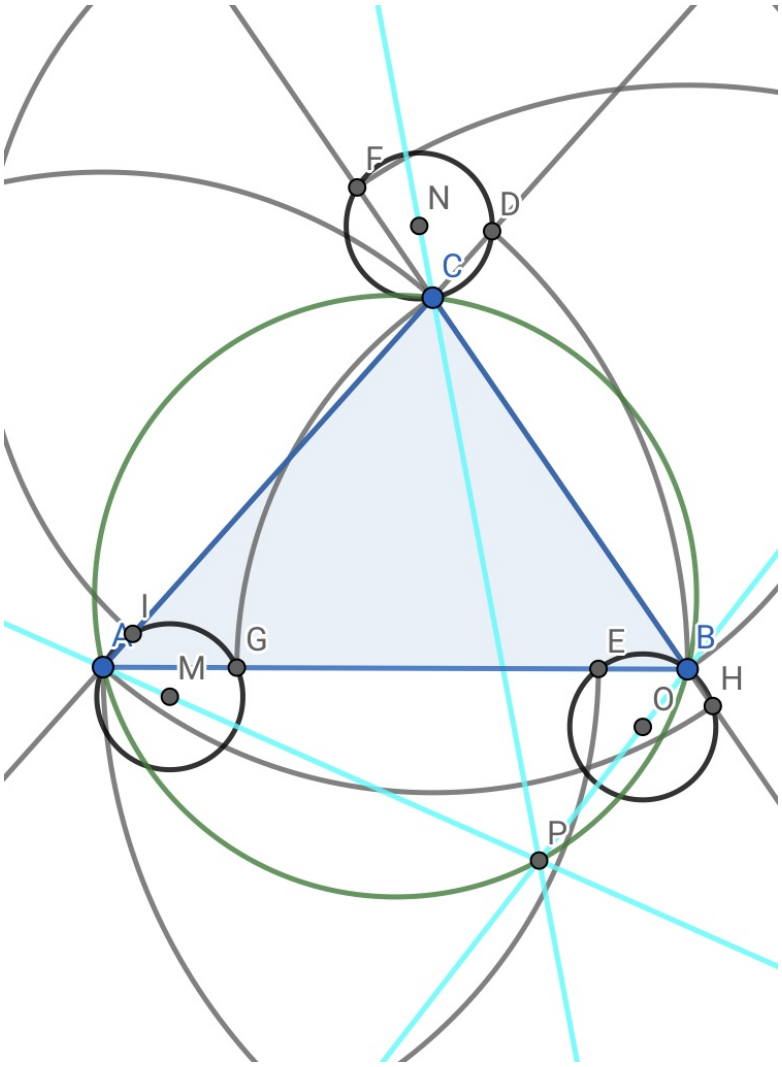

Figure 3. Circles AGI, CDF, and BEH are congruent and lines AM, BO, and CN are concurrent at $X(104)$ which lies on the circumcircle of ABC.

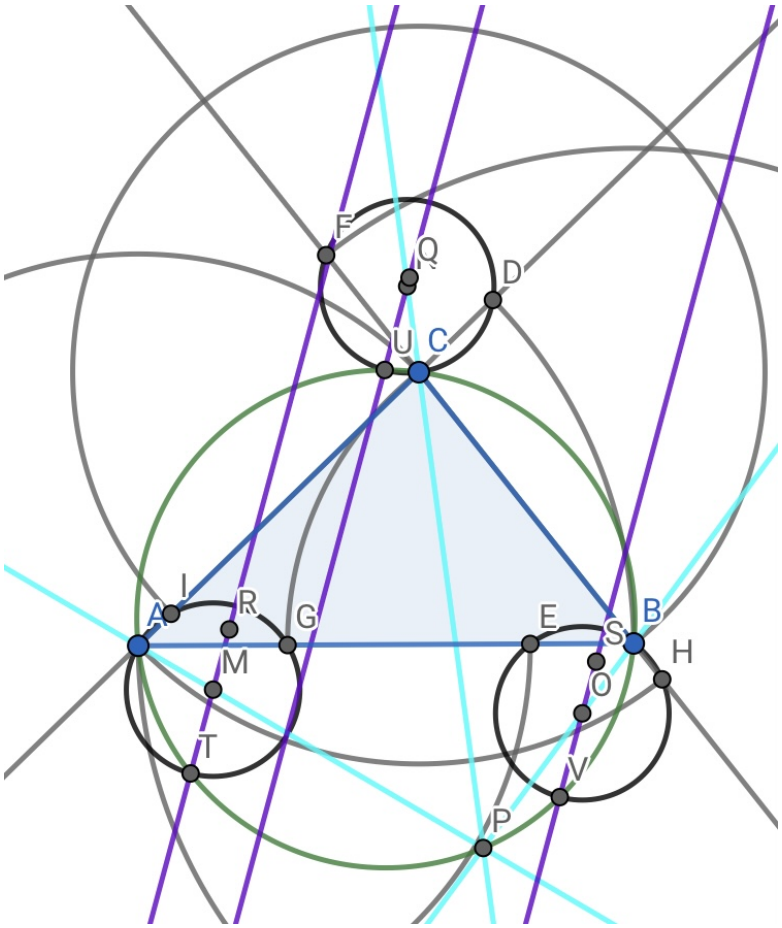

Figure 4. R, M, and T are collinear, N, Q, and U are collinear, and O, S, and V are collinear, and lines RMT, NQU, and OSV are parallel.