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The Midpoints of the Centroids of the Six Triangles Dissected by Concurrent Cevians are in Perspective with the Triangle

Benjamin Lee Warren[a] [2]
University of Binghamton, Mathematics,
4400 Vestel Pkwy E, Binghamton, NY 13902
e-mail: warrenbenjamin52@yahoo.com
web: https://orcid.org/0000-0002-2829-6953

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1. Introduction

We prove here that if a triangle is dissected into six smaller triangles by three concurrent cevians, then the midpoints of the adjacent centroids of these triangles are in perspective with the triangle ABC.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point in the plane, not on a triangle side.
Let lines AB and CD intersect at E.
Let lines AC and BD intersect at F.
Let lines AD and BC intersect at G.
Let α be the centroid of ΔADE.
Let β be the centroid of ΔADF.
Let γ be the centroid of ΔCDF.
Let δ be the centroid of ΔCDG.
Let ε be the centroid of ΔBDG.
Let ζ be the centroid of ΔBDE.
Let T be the midpoint of α and β.
Let U be the midpoint of γ and δ.
Let V be the midpoint of ε and ζ.

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[2]Corresponding author

Let lines AB and CU intersect at η.
Let lines AC and BV intersect at θ.
Let lines BC and AT intersect at ι.
Then,
1. Lines AT, BV, and CU are concurrent, and
2. E, F, G, η, θ, and ι lie on a conic.

Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $((x_C * y_D - x_D * y_C)/(y_D - y_C), 0)$.
F has coordinates $((x_C * y_D)/(x_C * y_D - x_D * y_C + y_C), (y_C * y_D)/(x_C * y_D - x_D * y_C + y_C))$.
G has coordinates $((x_D * y_C)/(-x_C * y_D + y_D + x_D * y_C), (y_C * y_D)/(-x_C * y_D + y_D + x_D * y_C))$.
α has coordinates
$(((x_C * y_D - x_D * y_C)/(y_D - y_C) + x_D)/3, y_D/3)$.
β has coordinates
$(((x_C*y_D)/(x_C*y_D-x_D*y_C+y_C)+x_D)/3, ((y_C*y_D)/(x_C*y_D-x_D*y_C+y_C)+y_D)/3)$.
γ has coordinates
$(((x_C * y_D)/(x_C * y_D - x_D * y_C + y_C) + x_D + x_C)/3, ((y_C * y_D)/(x_C * y_D - x_D * y_C + y_C) + y_D + y_C)/3)$.
δ has coordinates
$((x_C+x_D+((x_D*y_C)/(-x_C*y_D+y_D+x_D*y_C)))/3, (y_C+y_D+((y_C*y_D)/(-x_C*y_D+y_D+x_D*y_C)))/3)$.
ε has coordinates
$((x_D+((x_D*y_C)/(-x_C*y_D+y_D+x_D*y_C))+1)/3, (y_D+((y_C*y_D)/(-x_C*y_D+y_D+x_D*y_C)))/3)$.
ζ has coordinates
$((x_D + ((x_C * y_D - x_D * y_C)/(y_D - y_C)) + 1)/3, y_D/3)$.
T has coordinates
$(((x_C * y_D)/(x_C * y_D - x_D * y_C + y_C) + x_D)/6 + ((x_C * y_D - x_D * y_C)/(y_D - y_C) + x_D)/6, ((y_C * y_D)/(x_C * y_D - x_D * y_C + y_C) + y_D)/6 + y_D/6)$.
U has coordinates
$(((x_C+x_D+((x_C*y_D)/(x_C*y_D-x_D*y_C+y_C)))/3+(x_C+x_D+((x_D*y_C)/(-x_C*y_D+y_D+x_D*y_C)))/3)/2, ((y_C + y_D + ((y_C * y_D)/(x_C * y_D - x_D * y_C + y_C)))/3 + (y_C + y_D + ((y_C * y_D)/(-x_C * y_D + y_D + x_D * y_C)))/3)/2)$.
V has coordinates
$(((x_D+((x_D*y_C)/(-x_C*y_D+y_D+x_D*y_C))+1)/3+(x_D+((x_C*y_D-x_D*y_C)/(y_D-y_C))+1)/3)/2, ((y_D + ((y_C * y_D)/(-x_C * y_D + y_D + x_D * y_C)))/3 + (y_D)/3)/2)$.
Line AT is given by the equation
$y = (x * (((y_C * y_D)/(x_C * y_D - x_D * y_C + y_C) + y_D)/6 + y_D/6))/(((x_C * y_D)/(x_C * y_D - x_D * y_C + y_C) + x_D)/6 + ((x_C * y_D - x_D * y_C)/(y_D - y_C) + x_D)/6)$.
Line BV is given by the equation
$y = ((x-1) * (((y_C * y_D)/(-x_C * y_D + y_D + x_D * y_C) + y_D)/3 + y_D/3))/(2 * ((((x_D * y_C)/(-x_C * y_D + y_D + x_D * y_C) + x_D + 1)/3 + ((x_C * y_D - x_D * y_C)/(y_D - y_C) + x_D + 1)/3)/2 - 1))$.
Line CU is given by the equation

$y = ((x-x_C)*((((y_C*y_D)/(x_C*y_D-x_D*y_C+y_C)+y_D+y_C)/3+((y_C*y_D)/(-x_C*y_D+y_D+x_D*y_C)+y_D+y_C)/3)/2-y_C))/((((x_C*y_D)/(x_C*y_D-x_D*y_C+y_C)+x_D+x_C)/3+((x_D*y_C)/(-x_C*y_D+y_D+x_D*y_C)+x_D+x_C)/3)/2-x_C)+y_C$.
η has coordinates
$(((2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(x_C*y_D-(x_D-1)*y_C)*(x_C*y_D-x_D*y_C))/(2*(x_C-1)*x_C*y_D^3-(2*((2*x_C-1)*x_D+x_C*(2*x_C-3))+3)*y_C*y_D^2+(2*(x_D*(x_D+4*x_C-3)-2*x_C)+3)*y_C^2*y_D-4*(x_D-1)*x_D*y_C^3), 0)$.
θ has coordinates
$((x_C*y_D*(y_D-y_C)*(2*(x_C-1)*y_D-(2*x_D+1)*y_C))/(2*(x_C-1)*x_C*y_D^3-(2*(2*x_C-1)*x_D+3*x_C^2-6*x_C+4)*y_C*y_D^2+(2*x_D^2+6*(x_C-1)*x_D-3*x_C+4)*y_C^2*y_D-3*(x_D-1)*x_D*y_C^3), (y_C*y_D*(y_D-y_C)*(2*(x_C-1)*y_D-(2*x_D+1)*y_C))/(2*(x_C-1)*x_C*y_D^3-(2*(2*x_C-1)*x_D+3*x_C^2-6*x_C+4)*y_C*y_D^2+(2*x_D^2+6*(x_C-1)*x_D-3*x_C+4)*y_C^2*y_D-3*(x_D-1)*x_D*y_C^3))$.
ι has coordinates
$(-(y_C*(x_C*(2*x_D+x_C+1)*y_D^2-2*x_D*(x_D+2*x_C-1)*y_C*y_D+3*(x_D-1)*x_D*y_C^2))/(2*(x_C-1)*x_C*y_D^3-(2*(2*x_C-1)*x_D+3*x_C^2-4*x_C+3)*y_C*y_D^2+(2*x_D^2+2*(3*x_C-2)*x_D-3*x_C+3)*y_C^2*y_D-3*(x_D-1)*x_D*y_C^3), -(y_C*y_D*(y_D-y_C)*(2*x_C*y_D-(2*x_D-3)*y_C))/(2*(x_C-1)*x_C*y_D^3-(2*(2*x_C-1)*x_D+3*x_C^2-4*x_C+3)*y_C*y_D^2+(2*x_D^2+2*(3*x_C-2)*x_D-3*x_C+3)*y_C^2*y_D-3*(x_D-1)*x_D*y_C^3))$.
Consider the point W whose coordinates are given by
$(((2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(x_C*(2*x_D+x_C+1)*y_D^2-2*x_D*(x_D+2*x_C-1)*y_C*y_D+3*(x_D-1)*x_D*y_C^2))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3)), (y_D*(y_D-y_C)*(2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(2*x_C*y_D-(2*x_D-3)*y_C))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3)))$.
We verify that W lies on lines AT, BV, and CU since we calculate that
$((((2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(x_C*(2*x_D+x_C+1)*y_D^2-2*x_D*(x_D+2*x_C-1)*y_C*y_D+3*(x_D-1)*x_D*y_C^2))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3)))*(((y_C*y_D)/(x_C*y_D-x_D*y_C+y_C)+y_D)/6+y_D/6))/(((x_C*y_D)/(x_C*y_D-x_D*y_C+y_C)+x_D)/6+((x_C*y_D-x_D*y_C)/(y_D-y_C)+x_D)/6)-((y_D*(y_D-y_C)*(2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(2*x_C*y_D-(2*x_D-3)*y_C))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3))) = 0$,
$((((2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(x_C*(2*x_D+x_C+1)*y_D^2-2*x_D*(x_D+2*x_C-1)*y_C*y_D+3*(x_D-1)*x_D*y_C^2))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3))-1)*(((y_C*y_D)/(-x_C*y_D+y_D+x_D*y_C)+y_D)/3+y_D/3))/(2*((((x_D*y_C)/(-x_C*y_D+y_D+x_D*y_C)+x_D+1)/3+((x_C*y_D-x_D*y_C)/(y_D-y_C)+x_D+1)/3)/2-1))-(y_D*(y_D-y_C)*(2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(2*x_C*y_D-(2*x_D-3)*y_C))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3)) = 0$,
and
$((((2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(x_C*(2*x_D+x_C+1)*y_D^2-2*x_D*(x_D+2*x_C-1)*y_C*y_D+3*(x_D-1)*x_D*y_C^2))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3))-x_C)*((((y_C*y_D)/(x_C*y_D-x_D*y_C+y_C)+y_D+y_C)/3+((y_C*y_D)/(-x_C*y_D+y_D+x_D*y_C)+y_D+y_C)/3)/2-y_C))/((((x_C*y_D)/(x_C*y_D-$

$x_D*y_C+y_C)+x_D+x_C)/3+((x_D*y_C)/(-x_C*y_D+y_D+x_D*y_C)+x_D+x_C)/3)/2-x_C)+y_C-(y_D*(y_D-y_C)*(2*(x_C-1)*y_D-(2*x_D+1)*y_C)*(2*x_C*y_D-(2*x_D-3)*y_C))/(2*(5*(x_C-1)*x_C*y_D^3-(5*(2*x_C-1)*x_D+6*x_C^2-11*x_C+6)*y_C*y_D^2+(5*x_D^2+(12*x_C-11)*x_D-6*x_C+6)*y_C^2*y_D-6*(x_D-1)*x_D*y_C^3))=0.$
We then subsitute the above values into the following matrix to verify that

$$det\begin{bmatrix} x_E^2 & y_E^2 & x_E*y_E & x_E & y_E & 1 \\ x_F^2 & y_F^2 & x_F*y_F & x_F & y_F & 1 \\ x_G^2 & y_G^2 & x_G*y_G & x_G & y_G & 1 \\ x_\eta^2 & y_\eta^2 & x_\eta*y_\eta & x_\eta & y_\eta & 1 \\ x_\theta^2 & y_\theta^2 & x_\theta*y_\theta & x_\theta & y_\theta & 1 \\ x_\iota^2 & y_\iota^2 & x_\iota*y_\iota & x_\iota & y_\iota & 1 \end{bmatrix} = 0.$$

□

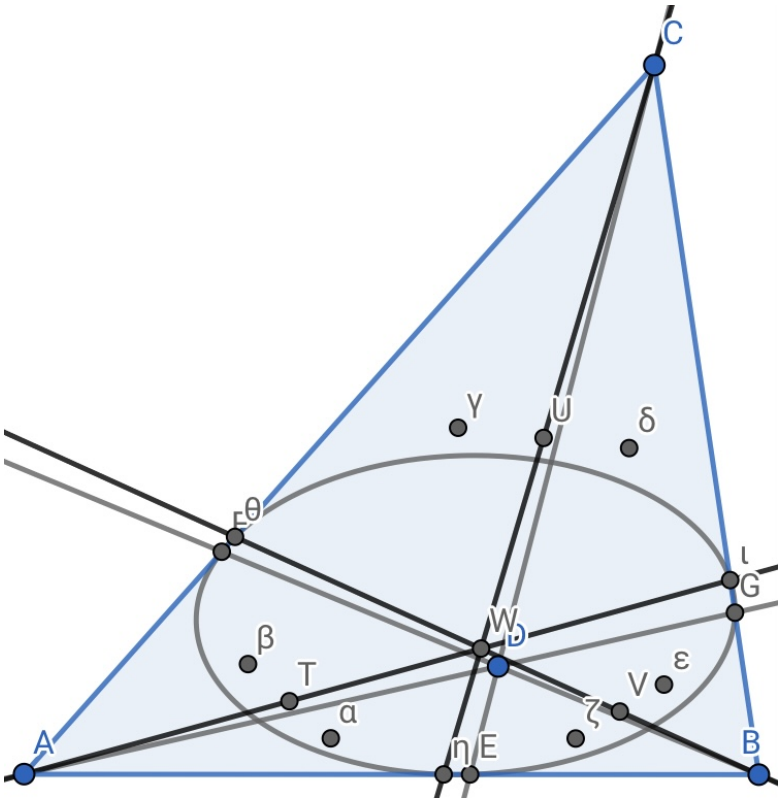

Figure 1. Illustration.