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The Bottema Point Lies on a Circle

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1. Introduction

We will demonstrate here that if two similar rectangles are erected on the sides of a triangle, then their centers, the Bottema point, the point where their circumcircles intersect, and the two points where the extended rectangle diagonals intersect the circumcircles are all concyclic.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let $ACFG$ form a rectangle, with F adjacent to A.
Let $BCHI$ form a rectangle similar to $ACFG$, with I adjacent to B.
Let lines AG and CF intersect at J.
Let lines BH and CI intersect at K.
Let L be the Bottema point (the midpoint of F and I).
Let circles $ACFG$ and $BCHI$ intersect at M.
Let line CIK intersect circle $ACFG$ at N.
Let line CFJ intersect circle $BCHI$ at O.
Then J, K, L, M, N, and O are concyclic.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
F has coordinates $(x_F, -(x_C * x_F)/y_C)$.
G has coordinates $(x_C + x_F, y_C - (x_C * x_F)/y_C)$.
H has coordinates $(x_C - x_F, y_C - ((1 - x_C) * x_F)/y_C)$.
I has coordinates $(1 - x_F, -((1 - x_C) * x_F)/y_C)$.
J has coordinates $((x_C + x_F)/2, (y_C - (x_C * x_F)/y_C)/2)$.
K has coordinates $((x_C - x_F + 1)/2, (y_C - ((1 - x_C) * x_F)/y_C)/2)$.
L has coordinates $(1/2, -x_F/(2 * y_C))$.
Circle $ACFG$ is given by the equation
$x^2 + (-x_F - x_C) * x - y * y_C + (x_C * x_F * y)/y_C + y^2 = 0$.
Circle $BCHI$ is given by the equation
$x^2 + (x_F - x_C - 1) * x - y * y_C + ((1 - x_C) * x_F * y)/y_C + y^2 - x_F + x_C = 0$.
M has coordinates
$(((x_F - x_C) * ((2 * x_F - 1) * y_C^2 - (2 * x_C - 1) * x_F^2))/((2 * x_F - 1)^2 * y_C^2 + (2 * x_C - 1)^2 * x_F^2), -(2 * x_F * (x_F - x_C) * (x_F + x_C - 1) * y_C)/((2 * x_F - 1)^2 * y_C^2 + (2 * x_C - 1)^2 * x_F^2))$.
N has coordinates
$(-((y_C^2 - x_F * (x_F + 2 * x_C - 2)) * ((x_F - 1) * y_C^2 + (x_C - 1) * x_C * x_F))/((y_C^2 + (x_C - 1)^2) * (y_C^2 + x_F^2)), (((x_F - 1) * y_C^2 + (x_C - 1) * x_C * x_F) * ((2 * x_F + x_C - 1) * y_C^2 - (x_C - 1) * x_F^2))/(y_C * (y_C^2 + (x_C - 1)^2) * (y_C^2 + x_F^2)))$.
O has coordinates
$((x_F * y_C^4 + (-x_F^3 + (2 * x_C + 2) * x_F^2 + (x_C^2 - 3 * x_C) * x_F + x_C^2) * y_C^2 + (x_C - x_C^2) * x_F^3 + (2 * x_C^3 - x_C^2) * x_F^2)/(y_C^4 + (x_F^2 + x_C^2) * y_C^2 + x_C^2 * x_F^2), (((x_F - 1) * y_C^2 + (x_C - 1) * x_C * x_F) * ((2 * x_F - x_C) * y_C^2 + x_C * x_F^2))/(y_C * (y_C^2 + x_C^2) * (y_C^2 + x_F^2)))$.
Now consider the circle whose equation is given by
$(x - (((2 * x_C + 3) * x_F - 2 * x_C - 1) * y_C^4 + x_F * (x_F * ((2 * x_C - 1) * x_F - 2 * x_C + 3) + (x_C - 1) * x_C * (2 * x_C + 3)) * y_C^2 + (x_C - 1) * x_C * (2 * x_C - 1) * x_F^3)/(4 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + (y - ((2 * x_F - 1) * y_C^4 + (2 * x_F^3 - 5 * x_F^2 + (2 * x_C^2 - 2 * x_C + 3) * x_F + x_C^2 - x_C) * y_C^2 + (2 * x_C^2 - 2 * x_C - 1) * x_F^3 - 3 * (x_C - 1) * x_C * x_F^2)/(4 * y_C * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 - (((x_C + x_F)/2 - (((2 * x_C + 3) * x_F - 2 * x_C - 1) * y_C^4 + x_F * (x_F * ((2 * x_C - 1) * x_F - 2 * x_C + 3) + (x_C - 1) * x_C * (2 * x_C + 3)) * y_C^2 + (x_C - 1) * x_C * (2 * x_C - 1) * x_F^3)/(4 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + ((y_C - (x_C * x_F)/y_C)/2 - ((2 * x_F - 1) * y_C^4 + (2 * x_F^3 - 5 * x_F^2 + (2 * x_C^2 - 2 * x_C + 3) * x_F + x_C^2 - x_C) * y_C^2 + (2 * x_C^2 - 2 * x_C - 1) * x_F^3 - 3 * (x_C - 1) * x_C * x_F^2)/(4 * y_C * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2) = 0$.
We verify that J, K, L, M, N, and O lie on this circle since we calculate that
$((x_C + x_F)/2 - (((2 * x_C + 3) * x_F - 2 * x_C - 1) * y_C^4 + x_F * (x_F * ((2 * x_C - 1) * x_F - 2 * x_C + 3) + (x_C - 1) * x_C * (2 * x_C + 3)) * y_C^2 + (x_C - 1) * x_C * (2 * x_C - 1) * x_F^3)/(4 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + ((y_C - (x_C * x_F)/y_C)/2 - ((2 * x_F - 1) * y_C^4 + (2 * x_F^3 - 5 * x_F^2 + (2 * x_C^2 - 2 * x_C + 3) * x_F + x_C^2 - x_C) * y_C^2 + (2 * x_C^2 - 2 * x_C - 1) * x_F^3 - 3 * (x_C - 1) * x_C * x_F^2)/(4 * y_C * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 - (((x_C + x_F)/2 - (((2 * x_C + 3) * x_F - 2 * x_C - 1) * y_C^4 + x_F * (x_F * ((2 * x_C - 1) * x_F - 2 * x_C + 3) + (x_C - 1) * x_C * (2 * x_C + 3)) * y_C^2 + (x_C - 1) * x_C * (2 * x_C - 1) * x_F^3)/(4 * y_C^2 * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2 + ((y_C - (x_C * x_F)/y_C)/2 - ((2 * x_F - 1) * y_C^4 + (2 * x_F^3 - 5 * x_F^2 + (2 * x_C^2 - 2 * x_C + 3) * x_F + x_C^2 - x_C) * y_C^2 + (2 * x_C^2 - 2 * x_C - 1) * x_F^3 - 3 * (x_C - 1) * x_C * x_F^2)/(4 * y_C * ((2 * x_F - 1) * y_C^2 + x_F * (x_F + 2 * (x_C - 1) * x_C))))^2) = 0$,
$((x_C - x_F + 1)/2 - (((2 * x_C + 3) * x_F - 2 * x_C - 1) * y_C^4 + x_F * (x_F * ((2 * x_C - 1) * x_F - 2 *$

$$x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((y_C-((1-x_C)*x_F)/y_C)/2-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2-(((x_C+x_F)/2-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2)=0,$$

$$(1/2-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+(-x_F/(2*y_C)-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2-(((x_C+x_F)/2-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2)=0,$$

$$(((x_F-x_C)*((2*x_F-1)*y_C^2-(2*x_C-1)*x_F^2))/((2*x_F-1)^2*y_C^2+(2*x_C-1)^2*x_F^2)-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+(-(2*x_F*(x_F-x_C)*(x_F+x_C-1)*y_C)/((2*x_F-1)^2*y_C^2+(2*x_C-1)^2*x_F^2)-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2-(((x_C+x_F)/2-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2)=0$$

$$(-((y_C^2-x_F*(x_F+2*x_C-2))*((x_F-1)*y_C^2+(x_C-1)*x_C*x_F))/((y_C^2+(x_C-1)^2)*(y_C^2+x_F^2))-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((((x_F-1)*y_C^2+(x_C-1)*x_C*x_F)*((2*x_F+x_C-1)*y_C^2-(x_C-1)*x_F^2))/(y_C*(y_C^2+(x_C-1)^2)*(y_C^2+x_F^2))-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2-(((x_C+x_F)/2-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2)=0,$$

and

$$((x_F*y_C^4+(-x_F^3+(2*x_C+2)*x_F^2+(x_C^2-3*x_C)*x_F+x_C^2)*y_C^2+(x_C-x_C^2)*x_F^3+(2*x_C^3-x_C^2)*x_F^2)/(y_C^4+(x_F^2+x_C^2)*y_C^2+x_C^2*x_F^2)-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((((x_F-1)*y_C^2+(x_C-1)*x_C*x_F)*((2*x_F-x_C)*y_C^2+x_C*x_F^2))/(y_C*(y_C^2+x_C^2)*(y_C^2+x_F^2))-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*$$

$x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2-(((x_C+x_F)/2-(((2*x_C+3)*x_F-2*x_C-1)*y_C^4+x_F*(x_F*((2*x_C-1)*x_F-2*x_C+3)+(x_C-1)*x_C*(2*x_C+3))*y_C^2+(x_C-1)*x_C*(2*x_C-1)*x_F^3)/(4*y_C^2*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2+((y_C-(x_C*x_F)/y_C)/2-((2*x_F-1)*y_C^4+(2*x_F^3-5*x_F^2+(2*x_C^2-2*x_C+3)*x_F+x_C^2-x_C)*y_C^2+(2*x_C^2-2*x_C-1)*x_F^3-3*(x_C-1)*x_C*x_F^2)/(4*y_C*((2*x_F-1)*y_C^2+x_F*(x_F+2*(x_C-1)*x_C))))^2) = 0$
as desired. □

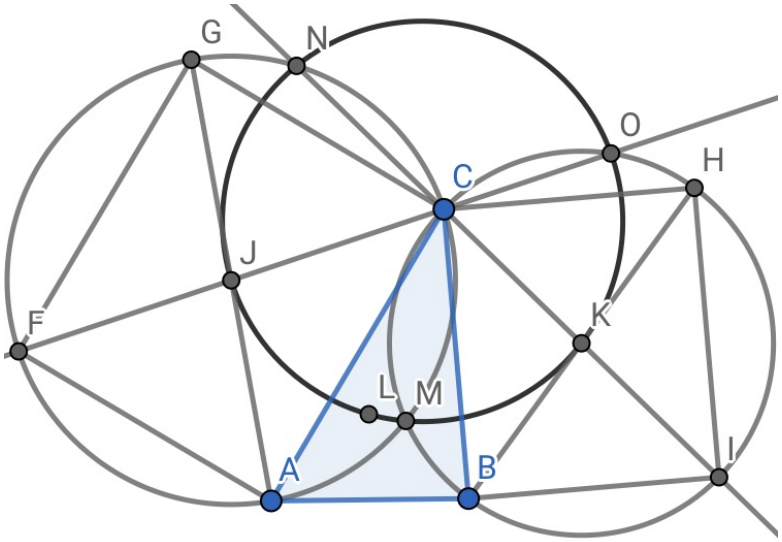

FIGURE 1. Illustration.

REFERENCES

[1] https://www.cut-the-knot.org/m/Geometry/GeneralBottema.shtml