International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 9, 2024, pp. 370-373.
Received August 27, 2024. Published on-line December 31, 2024.
web: http://www.journal-1.eu/
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The Line Passing Through the Centroid and the Cevian Point

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Mathematics Subject Classification (2010). 51N20.

1. Introduction

We prove here the existence of a point O which lies on the line between the centroid N and the cevian point D which satisfies $DN : NO = 2 : 1$.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point in the plane.
Construct three lines through each of the three vertices which are each parallel to the cevians, and let these lines intersect at points H, I, J, K, L, M, P, Q, and R.
Let N be the centroid of A, B, and C.
Then,
1. H, I, J, K, L, and M all lie on the same conic and A, B, C, P, Q, and R all lie on the same conic,
2. The midpoints of HK, IL, JM, AR, BP, and CQ are all the same and denoted as O,
3. D, N, and O are collinear with $DN : NO = 2 : 1$.

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Proof. Without loss of generality, we will work in Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordinates $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
Equation 1 is given by $y = (y_D * (x - x_C))/x_D + y_C$.
Equation 2 is given by $y = (y_D * (x - 1))/x_D$.
Equation 3 is given by $y = ((y_C - y_D) * x)/(x_C - x_D)$.
Equation 4 is given by $y = ((y_C - y_D) * (x - 1))/(x_C - x_D)$.
Equation 5 is given by $y = (y_D * (x - x_C))/(x_D - 1) + y_C$.
Equation 6 is given by $y = (y_D * x)/(x_D - 1)$.
H has coordinates
$((x_C * y_D - x_D * y_D)/(x_C * y_D - x_D * y_C), (y_D * (y_C - y_D))/(x_C * y_D - x_D * y_C))$.
I has coordinates
$((y_C - y_D - x_D * y_C + x_D * y_D)/(y_C - y_D + x_C * y_D - x_D * y_C), -(y_D * (y_C - y_D))/(y_C - y_D + x_C * y_D - x_D * y_C))$.
J has coordinates
$((y_D + x_D * y_C - x_D * y_D - x_D^2 * y_C + x_C * x_D * y_D)/y_D, y_C - y_D + x_C * y_D - x_D * y_C)$.
K has coordinates
$(-(x_D * y_C - x_D * y_D - x_C^2 * y_D - x_D^2 * y_C + x_C * x_D * y_C + x_C * x_D * y_D)/(x_C * y_D - x_D * y_C), -((y_C - y_D) * (y_D - x_C * y_D + x_D * y_C))/(x_C * y_D - x_D * y_C))$.
L has coordinates
$((x_C * y_C - x_D * y_C + x_C^2 * y_D + x_D^2 * y_C - x_C * x_D * y_C - x_C * x_D * y_D)/(y_C - y_D + x_C * y_D - x_D * y_C), ((y_C - y_D) * (y_C + x_C * y_D - x_D * y_C))/(y_C - y_D + x_C * y_D - x_D * y_C))$.
M has coordinates
$(-((x_C * y_D - x_D * y_C) * (x_D - 1))/y_D, x_D * y_C - x_C * y_D)$.
N has coordinates $((x_C + 1)/3, y_C/3)$.
P has coordinates
$(x_C - x_D, y_C - y_D)$.
Q has coordinates
$(1 - x_D, -y_D)$.
R has coordinates
$(x_C - x_D + 1, y_C - y_D)$.
Consider the point O whose coordinates are given by $((x_C - x_D + 1)/2, (y_C - y_D)/2)$.
We verify that O is the midpoint of H and K since we calculate
$(1/2) * ((x_C * y_D - x_D * y_D)/(x_C * y_D - x_D * y_C) - (x_D * y_C - x_D * y_D - x_C^2 * y_D - x_D^2 * y_C + x_C * x_D * y_C + x_C * x_D * y_D)/(x_C * y_D - x_D * y_C)) - (x_C - x_D + 1)/2 = 0$
and
$(1/2) * ((y_D * (y_C - y_D))/(x_C * y_D - x_D * y_C) - ((y_C - y_D) * (y_D - x_C * y_D + x_D * y_C))/(x_C * y_D - x_D * y_C)) - (y_C - y_D)/2 = 0$.
We verify that O is the midpoint of I and L since we calculate
$(1/2) * ((y_C - y_D - x_D * y_C + x_D * y_D)/(y_C - y_D + x_C * y_D - x_D * y_C) + (x_C * y_C - x_D * y_C + x_C^2 * y_D + x_D^2 * y_C - x_C * x_D * y_C - x_C * x_D * y_D)/(y_C - y_D + x_C * y_D - x_D * y_C)) - (x_C - x_D + 1)/2 = 0$
and
$(1/2) * (-(y_D * (y_C - y_D))/(y_C - y_D + x_C * y_D - x_D * y_C) + ((y_C - y_D) * (y_C + x_C * y_D - x_D * y_C))/(y_C - y_D + x_C * y_D - x_D * y_C)) - (y_C - y_D)/2 = 0$.
We verify that O is the midpoint of J and M since we calculate
$(1/2) * ((y_D + x_D * y_C - x_D * y_D - x_D^2 * y_C + x_C * x_D * y_D)/y_D - ((x_C * y_D - x_D *$

$y_C) * (x_D - 1))/y_D) - (x_C - x_D + 1)/2 = 0$
and
$(1/2) * (y_C - y_D + x_C * y_D - x_D * y_C + x_D * y_C - x_C * y_D) - (y_C - y_D)/2 = 0.$
One can clearly see by the coordinates that O is the midpoint of A and R, B and P, and C and Q.
One can verify by substitution that

$$det \begin{bmatrix} x_H^2 & y_H^2 & x_H * y_H & x_H & y_H & 1 \\ x_I^2 & y_I^2 & x_I * y_I & x_I & y_I & 1 \\ x_J^2 & y_J^2 & x_J * y_J & x_J & y_J & 1 \\ x_K^2 & y_K^2 & x_K * y_K & x_K & y_K & 1 \\ x_L^2 & y_L^2 & x_L * y_L & x_L & y_L & 1 \\ x_M^2 & y_M^2 & x_M * y_M & x_M & y_M & 1 \end{bmatrix} = 0$$

and

$$det \begin{bmatrix} x_A^2 & y_A^2 & x_A * y_A & x_A & y_A & 1 \\ x_B^2 & y_B^2 & x_B * y_B & x_B & y_B & 1 \\ x_C^2 & y_C^2 & x_C * y_C & x_C & y_C & 1 \\ x_P^2 & y_P^2 & x_P * y_P & x_P & y_P & 1 \\ x_Q^2 & y_Q^2 & x_Q * y_Q & x_Q & y_Q & 1 \\ x_R^2 & y_R^2 & x_R * y_R & x_R & y_R & 1 \end{bmatrix} = 0.$$

Now consider the line given by the equation
$y = ((y_D - y_C/3) * (x - x_D))/(x_D - (x_C + 1)/3) + y_D$. We verify that D, N, and O lie on this line since we calculate that
$((y_D - y_C/3) * (x_D - x_D))/(x_D - (x_C + 1)/3) + y_D - y_D = 0,$
$((y_D - y_C/3) * ((x_C + 1)/3 - x_D))/(x_D - (x_C + 1)/3) + y_D - y_C/3 = 0,$
and
$((y_D - y_C/3) * ((x_C - x_D + 1)/2 - x_D))/(x_D - (x_C + 1)/3) + y_D - (y_C - y_D)/2 = 0.$
We then calculate that $DN : NO = (x_D - x_N)/(x_N - x_O)$
$= (x_D - (x_C + 1)/3)/((x_C + 1)/3 - (x_C - x_D + 1)/2) = 2 : 1.$ □

References

[1] Weisstein, Eric W. "Cevian Point." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Orthocenter.html
[2] Weisstein, Eric W. "Kimberling Center." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Orthocenter.html

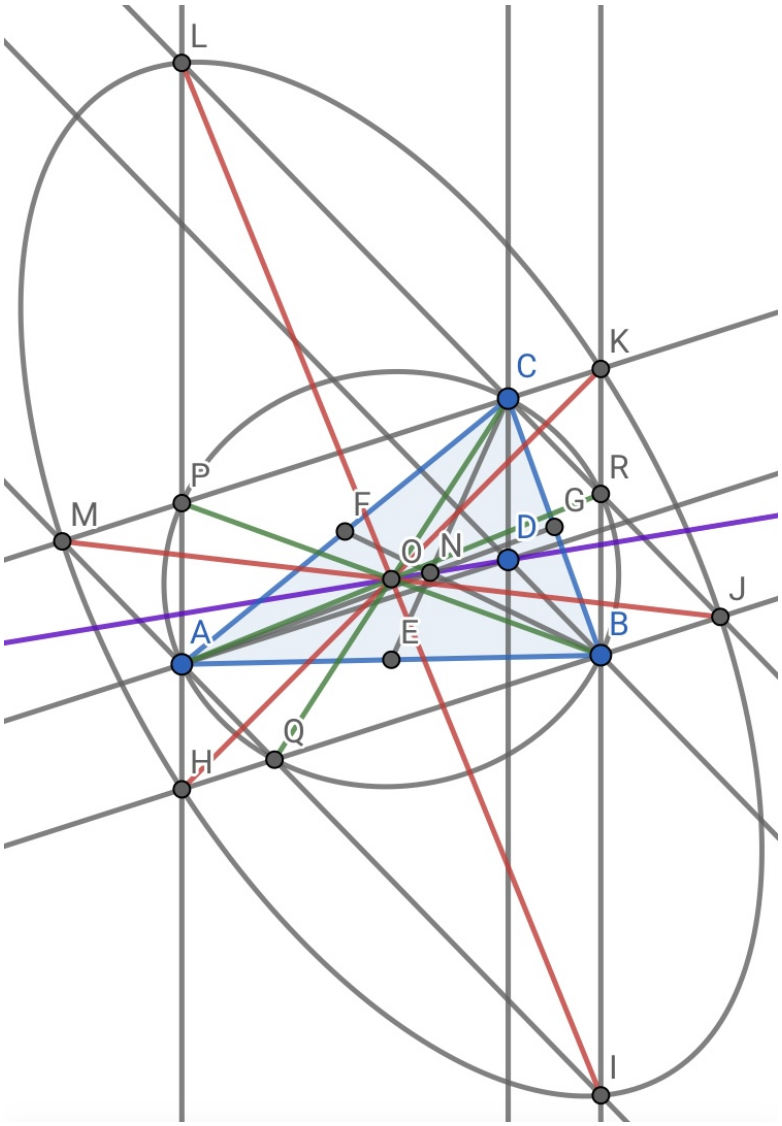

FIGURE 1. Illustration.