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Five Concurrent Circles Formed from Two Cevians

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1. Introduction

Here we will give an analytical proof of the existence of the Steiner circle.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be a point on line AC.
Let E be a point on line BC.
Let lines AE and BD intersect at point F.
Let G be the circumcenter of A, D, and F.
Let H be the circumcenter of A, C, and E.
Let I be the cirumcenter of B, C, and D.
Let J be the cirumcenter of B, E, and F.
Then,
1. Circles ADF, ACE, BCD, and BEF are concurrent at point K, and
2. Points G, H, I, J, and K are concyclic.

Proof. Without loss of generality, we will use Cartesian coordinates.
A has coordinates $(0, 0)$.
B has coordintes $(1, 0)$.
C has coordinates (x_C, y_C).
D has coordinates $(x_D, (x_D * y_C)/x_C)$.
E has coordinates $(x_E, ((x_E - 1) * y_C)/(x_C - 1))$.

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F has coordinates
$(-(x_D * x_E * (x_C - 1))/(x_C - x_C * x_D - x_C * x_E + x_D * x_E), -(x_D * y_C * (x_E - 1))/(x_C - x_C * x_D - x_C * x_E + x_D * x_E))$.
G has coordinates
$((x_D * (x_D * (x_E - 1) * y_C^2 + x_C * (x_C * (x_D - 2) + 1) * x_E - x_C^2 * x_D + x_C^2))/(2 * x_C * ((x_D - x_C) * x_E - x_C * x_D + x_C)), -(x_D * ((((x_C - 1) * x_D + x_C) * x_E - x_C) * y_C^2 + (x_C - 1) * x_C^2 * (x_D - 1) * x_E))/(2 * x_C * ((x_D - x_C) * x_E - x_C * x_D + x_C) * y_C))$.
H has coordinates
$((x_C - (x_C - 1) * x_E)/2 - ((x_E - 1) * y_C^2)/(2 * (x_C - 1)), ((x_C * x_E - 1) * y_C)/(2 * (x_C - 1)) + ((x_C - 1) * x_C * x_E)/(2 * y_C))$.
I has coordinates
$((x_D * x_C^2 + x_C + x_D * y_C^2)/(2 * x_C), -(((x_C - 1) * x_D - x_C) * y_C^2 + (x_C - 1) * x_C^2 * x_D - x_C^3 + x_C^2)/(2 * x_C * y_C))$.
J has coordinates
$(-(x_D * (x_E - 1)^2 * y_C^2 + (x_C - 1)^2 * x_D * x_E^2 - (x_C - 1) * (x_D - x_C) * x_E + (x_C - 1) * x_C * x_D - x_C^2 + x_C)/(2 * (x_C - 1) * ((x_D - x_C) * x_E - x_C * x_D + x_C)), ((x_E - 1) * ((x_C * (x_D - 1) * x_E - (2 * x_C - 1) * x_D + x_C) * y_C^2 + (x_C - 1)^2 * x_C * (x_D - 1) * x_E))/(2 * (x_C - 1) * ((x_D - x_C) * x_E - x_C * x_D + x_C) * y_C))$.
Circle ADF is given by the equation
$(x - (x_D * (x_D * (x_E - 1) * y_C^2 + x_C * (x_C * (x_D - 2) + 1) * x_E - x_C^2 * x_D + x_C^2))/(2 * x_C * ((x_D - x_C) * x_E - x_C * x_D + x_C)))^2 + (y + (x_D * ((((x_C - 1) * x_D + x_C) * x_E - x_C) * y_C^2 + (x_C - 1) * x_C^2 * (x_D - 1) * x_E))/(2 * x_C * ((x_D - x_C) * x_E - x_C * x_D + x_C) * y_C))^2 - (((x_D * (x_D * (x_E - 1) * y_C^2 + x_C * (x_C * (x_D - 2) + 1) * x_E - x_C^2 * x_D + x_C^2))/(2 * x_C * ((x_D - x_C) * x_E - x_C * x_D + x_C)))^2 + ((x_D * ((((x_C - 1) * x_D + x_C) * x_E - x_C) * y_C^2 + (x_C - 1) * x_C^2 * (x_D - 1) * x_E))/(2 * x_C * ((x_D - x_C) * x_E - x_C * x_D + x_C) * y_C))^2) = 0$.
Circle ACE is given by the equation
$(x - ((x_C - (x_C - 1) * x_E)/2 - ((x_E - 1) * y_C^2)/(2 * (x_C - 1))))^2 + (y - (((x_C * x_E - 1) * y_C)/(2 * (x_C - 1)) + ((x_C - 1) * x_C * x_E)/(2 * y_C)))^2 - ((-((x_C - (x_C - 1) * x_E)/2 - ((x_E - 1) * y_C^2)/(2 * (x_C - 1))))^2 + (-(((x_C * x_E - 1) * y_C)/(2 * (x_C - 1)) + ((x_C - 1) * x_C * x_E)/(2 * y_C)))^2) = 0$.
Circle BCD is given by the equation
$(x - (x_D * x_C^2 + x_C + x_D * y_C^2)/(2 * x_C))^2 + (y + (((x_C - 1) * x_D - x_C) * y_C^2 + (x_C - 1) * x_C^2 * x_D - x_C^3 + x_C^2)/(2 * x_C * y_C))^2 - ((1 - (x_D * x_C^2 + x_C + x_D * y_C^2)/(2 * x_C))^2 + ((((x_C - 1) * x_D - x_C) * y_C^2 + (x_C - 1) * x_C^2 * x_D - x_C^3 + x_C^2)/(2 * x_C * y_C))^2) = 0$.
Circle BEF is given by the equation
$(x + (x_D * (x_E - 1)^2 * y_C^2 + (x_C - 1)^2 * x_D * x_E^2 - (x_C - 1) * (x_D - x_C) * x_E + (x_C - 1) * x_C * x_D - x_C^2 + x_C)/(2 * (x_C - 1) * ((x_D - x_C) * x_E - x_C * x_D + x_C)))^2 + (y - ((x_E - 1) * ((x_C * (x_D - 1) * x_E - (2 * x_C - 1) * x_D + x_C) * y_C^2 + (x_C - 1)^2 * x_C * (x_D - 1) * x_E))/(2 * (x_C - 1) * ((x_D - x_C) * x_E - x_C * x_D + x_C) * y_C))^2 - ((1 + (x_D * (x_E - 1)^2 * y_C^2 + (x_C - 1)^2 * x_D * x_E^2 - (x_C - 1) * (x_D - x_C) * x_E + (x_C - 1) * x_C * x_D - x_C^2 + x_C)/(2 * (x_C - 1) * ((x_D - x_C) * x_E - x_C * x_D + x_C)))^2 + (-((x_E - 1) * ((x_C * (x_D - 1) * x_E - (2 * x_C - 1) * x_D + x_C) * y_C^2 + (x_C - 1)^2 * x_C * (x_D - 1) * x_E))/(2 * (x_C - 1) * ((x_D - x_C) * x_E - x_C * x_D + x_C) * y_C))^2) = 0$.
Now, consider the point K whose coordinates are given by
$((x_D * ((x_C^2 * x_E^2 + ((x_C - 1)^2 * x_D - x_C^2 - x_C) * x_E + x_C) * y_C^2 + (x_C - 1)^2 * x_C^2 * x_E^2 + (x_C - 1)^2 * x_C^2 * (x_D - 1) * x_E))/((x_C * x_E + x_C * x_D - x_D - x_C)^2 * y_C^2 + (x_C - 1)^2 * x_C^2 * x_E^2 + 2 * (x_C - 1)^2 * x_C^2 * (x_D - 1) * x_E + (x_C - 1)^2 * x_C^2 * x_D^2 - 2 * (x_C - 1)^2 * x_C^2 * x_D + x_C^4 - 2 * x_C^3 + x_C^2), (x_D * (x_E - 1) * y_C * ((x_C * x_E + (x_C - 1) * x_D - x_C) * y_C^2 + (x_C - 1)^2 * x_C * x_E + (x_C - 1) * x_C^2 * x_D - x_C^3 + x_C^2))/((x_C * x_E + x_C * x_D - x_D - x_C)^2 * y_C^2 + (x_C - 1)^2 * x_C^2 * x_E^2 + 2 * (x_C - 1)^2 * x_C^2 * (x_D - 1) * x_E + (x_C - 1)^2 * x_C^2 * x_D^2 - 2 * (x_C - 1)^2 * x_C^2 * x_D + x_C^4 - 2 * x_C^3 + x_C^2))$.

We verify that K lies on all the above circles since we calculate that
$((x_D*((x_C^2*x_E^2+((x_C-1)^2*x_D-x_C^2-x_C)*x_E+x_C)*y_C^2+(x_C-1)^2*x_C^2*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*x_E))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)-(x_D*(x_D*(x_E-1)*y_C^2+x_C*(x_C*(x_D-2)+1)*x_E-x_C^2*x_D+x_C^2))/(2*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)))^2+((x_D*(x_E-1)*y_C*((x_C*x_E+(x_C-1)*x_D-x_C)*y_C^2+(x_C-1)^2*x_C*x_E+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)+(x_D*((((x_C-1)*x_D+x_C)*x_E-x_C)*y_C^2+(x_C-1)*x_C^2*(x_D-1)*x_E))/(2*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C))^2-(((x_D*(x_D*(x_E-1)*y_C^2+x_C*(x_C*(x_D-2)+1)*x_E-x_C^2*x_D+x_C^2))/(2*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)))^2+((x_D*((((x_C-1)*x_D+x_C)*x_E-x_C)*y_C^2+(x_C-1)*x_C^2*(x_D-1)*x_E))/(2*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C))^2)=0,$
$((x_D*((x_C^2*x_E^2+((x_C-1)^2*x_D-x_C^2-x_C)*x_E+x_C)*y_C^2+(x_C-1)^2*x_C^2*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*x_E))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)-((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))))^2+((x_D*(x_E-1)*y_C*((x_C*x_E+(x_C-1)*x_D-x_C)*y_C^2+(x_C-1)^2*x_C*x_E+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)-(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)))^2-((-((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))))^2+(-(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)))^2)=0,$
$((x_D*((x_C^2*x_E^2+((x_C-1)^2*x_D-x_C^2-x_C)*x_E+x_C)*y_C^2+(x_C-1)^2*x_C^2*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*x_E))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)-(x_D*x_C^2+x_C+x_D*y_C^2)/(2*x_C))^2+((x_D*(x_E-1)*y_C*((x_C*x_E+(x_C-1)*x_D-x_C)*y_C^2+(x_C-1)^2*x_C*x_E+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)+(((x_C-1)*x_D-x_C)*y_C^2+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2)/(2*x_C*y_C))^2-((1-(x_D*x_C^2+x_C+x_D*y_C^2)/(2*x_C))^2+(((x_C-1)*x_D-x_C)*y_C^2+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2)/(2*x_C*y_C))^2)=0,$
and
$((x_D*((x_C^2*x_E^2+((x_C-1)^2*x_D-x_C^2-x_C)*x_E+x_C)*y_C^2+(x_C-1)^2*x_C^2*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*x_E))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)+(x_D*(x_E-1)^2*y_C^2+(x_C-1)^2*x_D*x_E^2-(x_C-1)*(x_D-x_C)*x_E+(x_C-1)*x_C*x_D-x_C^2+x_C)/(2*(x_C-1)*((x_D-x_C)*x_E-x_C*x_D+x_C)))^2+((x_D*(x_E-1)*y_C*((x_C*x_E+(x_C-1)*x_D-x_C)*y_C^2+(x_C-1)^2*x_C*x_E+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)-((x_E-1)*((x_C*(x_D-1)*x_E-(2*x_C-1)*x_D+x_C)*y_C^2+(x_C-1)^2*x_C*(x_D-1)*x_E))/(2*(x_C-1)*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C))^2-((1+(x_D*(x_E-1)^2*y_C^2+(x_C-1)^2*x_D*x_E^2-(x_C-1)*(x_D-x_C)*x_E+(x_C-1)*x_C*x_D-x_C^2+x_C)/(2*(x_C-1)*((x_D-x_C)*x_E-x_C*x_D+x_C)))^2+(-((x_E-1)*((x_C*(x_D-1)*x_E-(2*x_C-1)*x_D+x_C)*y_C^2+(x_C-1)^2*x_C*(x_D-1)*x_E))/(2*(x_C-1)*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C))^2)=0$
as desired.
Next, consider the circle given by the equation
$(x-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+$

$$2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(y-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2-(((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2)=0.$$

We verify that points G, H, I, J, and K lie on this circle since we calculate that

$$((x_D*(x_D*(x_E-1)*y_C^2+x_C*(x_C*(x_D-2)+1)*x_E-x_C^2*x_D+x_C^2))/(2*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(-(x_D*((((x_C-1)*x_D+x_C)*x_E-x_C)*y_C^2+(x_C-1)*x_C^2*(x_D-1)*x_E))/(2*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2+(((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2)=0,$$

$$((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2-(((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2)=0,$$

$$((x_D*x_C^2+x_C+x_D*y_C^2)/(2*x_C)-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(-(((x_C-1)*x_D-x_C)*y_C^2+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2)/(2*x_C*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2-(((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2)=0,$$

$$(-(x_D*(x_E-1)^2*y_C^2+(x_C-1)^2*x_D*x_E^2-(x_C-1)*(x_D-x_C)*x_E+(x_C-1)*x_C*x_D-$$

$$x_C^2+x_C)/(2*(x_C-1)*((x_D-x_C)*x_E-x_C*x_D+x_C))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_E-1)*((x_C*(x_D-1)*x_E-(2*x_C-1)*x_D+x_C)*y_C^2+(x_C-1)^2*x_C*(x_D-1)*x_E))/(2*(x_C-1)*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2-(((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_C*x_E-1)*y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2)=0,$$

and

$$((x_D*((x_C^2*x_E^2+((x_C-1)^2*x_D-x_C^2-x_C)*x_E+x_C)*y_C^2+(x_C-1)^2*x_C^2*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*x_E))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+((x_D*(x_E-1)*y_C*((x_C*x_E+(x_C-1)*x_D-x_C)*y_C^2+(x_C-1)^2*x_C*x_E+(x_C-1)*x_C^2*x_D-x_C^3+x_C^2))/((x_C*x_E+x_C*x_D-x_D-x_C)^2*y_C^2+(x_C-1)^2*x_C^2*x_E^2+2*(x_C-1)^2*x_C^2*(x_D-1)*x_E+(x_C-1)^2*x_C^2*x_D^2-2*(x_C-1)^2*x_C^2*x_D+x_C^4-2*x_C^3+x_C^2)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2-(((x_C-(x_C-1)*x_E)/2-((x_E-1)*y_C^2)/(2*(x_C-1))-(-((x_C*(x_C*x_D+x_D-x_C)*x_E^2+((x_C-2)*(x_C-1)*x_D^2-2*x_C*(x_C+1)*x_D+2*x_C^2)*x_E+(x_C-1)*x_D^2+x_C*(x_C+1)*x_D-x_C^2)*y_C^4+(x_C-1)*x_C*(2*x_C^2*x_D*x_E^2-x_D*x_E^2-2*x_C^2*x_E^2+x_C*x_E^2+2*x_C^2*x_D^2*x_E-4*x_C*x_D^2*x_E+x_D^2*x_E-4*x_C^2*x_D*x_E+4*x_C*x_D*x_E-2*x_D*x_E+3*x_C^2*x_E+x_C*x_D^2+x_C^2*x_D-x_C^2-x_C)*y_C^2+(x_C-1)^3*x_C^3*(x_D-1)*x_E^2+(x_C-1)^3*x_C^3*(x_D-1)^2*x_E)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C^2)))^2+(((x_C*x_E-1)*$$

$y_C)/(2*(x_C-1))+((x_C-1)*x_C*x_E)/(2*y_C)-(-(x_D*(x_E-1)*(x_C*x_E+x_C*x_D-x_D-x_C)*y_C^4+(x_C*((2*x_C^2-4*x_C+1)*x_D+x_C)*x_E^2+((x_C-1)*(2*x_C^2-1)*x_D^2-2*x_C*(2*x_C^2-4*x_C+1)*x_D+x_C^3-3*x_C^2)*x_E-(x_C-1)*x_C*(2*x_C-1)*x_D^2+x_C*(3*x_C^2-6*x_C+2)*x_D-x_C^3+2*x_C^2)*y_C^2+(x_C-1)^2*x_C^2*((x_C-2)*x_D+1)*x_E^2+(x_C-1)^2*x_C^2*(x_D-1)*((x_C+1)*x_D-x_C+1)*x_E-(x_C-1)^2*x_C^3*x_D^2+2*(x_C-1)^2*x_C^3*x_D-x_C^5+2*x_C^4-x_C^3)/(4*(x_C-1)*x_C*((x_D-x_C)*x_E-x_C*x_D+x_C)*y_C)))^2) = 0$
as desired. □

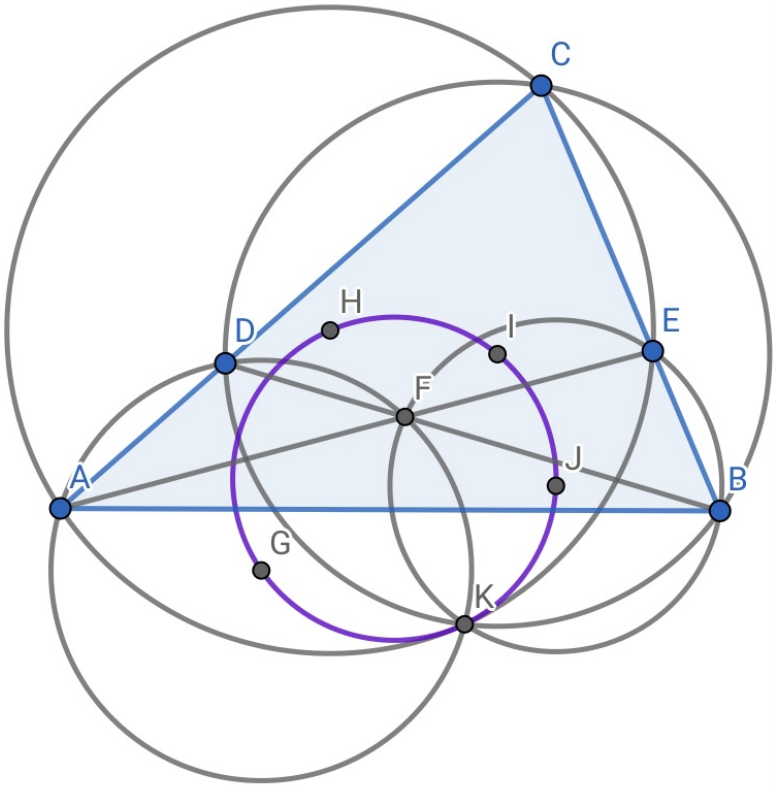

Figure 1. Illustration.