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Circles Centered at Vertices of Triangles

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1. Introduction

Pick a vertex of a given triangle. Centered at that vertex, draw two circles that respectively pass through the other two triangle vertices. Do this for all the vertices, making six circles in total. In [1], a phenomenon was discussed involving these circles. It was shown that when a certain three of them intersect at a point, the other three intersect at another point. Here, we construct these circles again and derive some results which generalize this phenomenon, and also identify some points in the construction that turn out to be collinear.

2. Preliminaries

Let A, B, and C be the vertices of a given triangle.
Let $\mathcal{O}AB$ denote the circle that is centered at A and which passes through B.
Let $\mathcal{O}BA$ denote the circle that is centered at B and which passes through A.
Let $\mathcal{O}AC$ denote the circle that is centered at A and which passes through C.
Let $\mathcal{O}CA$ denote the circle that is centered at C and which passes through A.
Let $\mathcal{O}BC$ denote the circle that is centered at B and which passes through C.
Let $\mathcal{O}CB$ denote the circle that is centered at C and which passes through B.
Let $\mathcal{O}AC$ and $\mathcal{O}CA$ intersect at D and G.
Let $\mathcal{O}BC$ and $\mathcal{O}CB$ intersect at E and F.
Let $\mathcal{O}AC$ and $\mathcal{O}BC$ intersect at C and H.
Let $\mathcal{O}AB$ and $\mathcal{O}BA$ intersect at I and K.
Let $\mathcal{O}CA$ and $\mathcal{O}BA$ intersect at A and J.

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Let $\mathcal{O}AB$ and $\mathcal{O}CB$ intersect at B and L.

3. Results

Theorem 3.1. $|AB||GH| = |AC||KL|$.

Theorem 3.2. $|AB||FH| = |BC||IJ|$.

Theorem 3.3. $\angle EHG = \frac{\pi}{3}$.

Theorem 3.4. *The line that bisects $\angle EHG$ passes through C.*

Theorem 3.5. *D, E, and H are collinear.*

Theorem 3.6. *F, G, and H are collinear.*

Theorem 3.7. *E, I, and L are collinear.*

Theorem 3.8. *D, J, and K are collinear.*

Theorem 3.9. *G, I, and J are collinear.*

Theorem 3.10. *F, K, and L are collinear.*

Proof. For the first two results, we set up the following.
Without loss of generality, we take the cartesian coordinates of A, B, and C to be respectively given by $(0,0), (x_b, 0)$, and (x_c, y_c).
$\mathcal{O}AC$ is given by the equation $x^2 + y^2 = x_c^2 + y_c^2$.
$\mathcal{O}CA$ is given by the equation $(x - x_c)^2 + (y - y_c)^2 = x_c^2 + y_c^2$.
$\mathcal{O}BC$ is given by the equation
$(x - x_b)^2 + y^2 = (x_b - x_c)^2 + y_c^2$.
$\mathcal{O}AB$ is given by the equation
$x^2 + y^2 = x_b^2$.
$\mathcal{O}BA$ is given by the equation
$(x - x_b)^2 + y^2 = x_b^2$.
$\mathcal{O}CB$ is given by the equation
$(x - x_c)^2 + (y - y_c)^2 = (x_b - x_c)^2 + y_c^2$.
$\mathcal{O}BC$ is given by the equation
$(x - x_b)^2 + y^2 = (x_b - x_c)^2 + y_c^2$.
The coordinates of F are given by $(\frac{1}{2}(x_b + x_c + y_c\sqrt{3}), \frac{1}{2}(y_c + \sqrt{3}(x_b - x_c)))$.
The coordinates of G are given by
$(\frac{1}{2}(x_c + y_c\sqrt{3}), \frac{1}{2}(-x_c\sqrt{3} + y_c))$.
The coordinates of H are given by $(x_c, -y_c)$.
The coordinates of K are given by $(\frac{1}{2}x_b, \frac{x_b\sqrt{3}}{2})$.
The coordinates of I are given by $(\frac{1}{2}x_b, -\frac{x_b\sqrt{3}}{2})$.
The coordinates of L are given by $(\frac{x_b(x_c^2 - y_c^2)}{x_c^2 + y_c^2}, \frac{2x_b x_c y_c}{x_c^2 + y_c^2})$.
The coordinates of J are given by $(\frac{2x_b y_c^2}{x_b^2 - 2x_b x_c + x_c^2 + y_c^2}, \frac{2x_b y_c (x_b - x_c)}{x_b^2 - 2x_b x_c + x_c^2 + y_c^2})$.

1. $(\frac{|GH|}{|KL|})^2 = \frac{(\frac{1}{2}(x_c + y_c\sqrt{3}) - x_c)^2 + (\frac{1}{2}(-x_c\sqrt{3} + y_c) + y_c)^2}{(\frac{1}{2}x_b - \frac{x_b(x_c^2 - y_c^2)}{x_c^2 + y_c^2})^2 + (\frac{x_b\sqrt{3}}{2} - \frac{2x_b x_c y_c}{x_c^2 + y_c^2})^2}$

$= \frac{x_c^2 + y_c^2}{x_b^2} \iff |GH| = \frac{|KL|\sqrt{x_c^2 + y_c^2}}{x_b} \iff |AB||GH| = |AC||KL|.$

2. $(\frac{|FH|}{|IJ|})^2 = \frac{(\frac{1}{2}(x_b + x_c + y_c\sqrt{3}) - x_c)^2 + (\frac{1}{2}(y_c + \sqrt{3}(x_b - x_c)) + y_c)^2}{(\frac{1}{2}x_b - \frac{2x_b y_c^2}{x_b^2 - 2x_b x_c + x_c^2 + y_c^2})^2 + (-\frac{x_b\sqrt{3}}{2} - (\frac{2x_b y_c (x_b - x_c)}{x_b^2 - 2x_b x_c + x_c^2 + y_c^2}))^2}$

$= \frac{y_c^2+(x_c-x_b)^2}{x_b^2} \iff \frac{|FH|}{|IJ|} = \frac{\sqrt{y_c^2+(x_c-x_b)^2}}{x_b} \iff |AB||FH| = |BC||IJ|$.
Note that the previous two results directly generalize the two results in [1].
For the rest of the results, we set up the following.
Without loss of generality, we take the cartesian coordinates of A, B, and C to be respectively given by $(0,0), (1,0)$, and (x_c, y_c).
$\mathcal{O}AB$ is given by the equation $x^2 + y^2 = 1$.
$\mathcal{O}BA$ is given by the equation $(x-1)^2 + y^2 = 1$.
$\mathcal{O}AC$ is given by the equation $x^2 + y^2 = x_c^2 + y_c^2$.
$\mathcal{O}CA$ is given by the equation $(x-x_c)^2 + (y-y_c)^2 = x_c^2 + y_c^2$.
$\mathcal{O}BC$ is given by the equation $(x-1)^2 + y^2 = (x_c-1)^2 + y_c^2$.
$\mathcal{O}CB$ is given by the equation $(x-x_c)^2 + (y-y_c)^2 = (x_c-1)^2 + y_c^2$.
Therefore the coordinates of D are given by $(\frac{1}{2}(x_c - y_c\sqrt{3}), \frac{1}{2}(x_c\sqrt{3} + y_c))$,
and the coordinates of G are given by $(\frac{1}{2}(x_c + y_c\sqrt{3}), \frac{1}{2}(-x_c\sqrt{3} + y_c))$.
The coordinates of E are given by $(\frac{1}{2}(x_c - y_c\sqrt{3} + 1), \frac{1}{2}(y_c + \sqrt{3}(x_c - 1)))$,
and the coordinates of F are given by $(\frac{1}{2}(x_c + y_c\sqrt{3} + 1), \frac{1}{2}(y_c + \sqrt{3}(1 - x_c)))$.
The coordinates of H are given by $(x_c, -y_c)$.
The coordinates of I are given by $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$,
and the coordinates of K are given by $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.
The coordinates of L are given by $(\frac{x_c^2-y_c^2}{x_c^2+y_c^2}, \frac{2x_cy_c}{x_c^2+y_c^2})$.
The coordinates of J are given by $(\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1}, \frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1})$.
3. $\arctan(\frac{x_H-x_E}{y_E-y_H}) = \arctan(\frac{x_c-\frac{1}{2}(x_c-y_c\sqrt{3}+1)}{\frac{1}{2}(y_c+\sqrt{3}(x_c-1))+y_c}) = \arctan(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}$.
Also, $\arctan(\frac{x_G-x_H}{y_G-y_H}) = \arctan(\frac{\frac{1}{2}(x_c+y_c\sqrt{3})-x_c}{\frac{1}{2}(-x_c\sqrt{3}+y_c)+y_c}) = \arctan(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}$. Putting these together gives $\angle EHG = \frac{\pi}{3}$ as desired.
4. The slope of the line passing through E and H is $(\frac{\frac{1}{2}(y_c+\sqrt{3}(x_c-1))+y_c}{\frac{1}{2}(x_c-y_c\sqrt{3}+1)-x_c})$. The slope of the line passing through G and H is $(\frac{\frac{1}{2}(-x_c\sqrt{3}+y_c)+y_c}{\frac{1}{2}(x_c+y_c\sqrt{3})-x_c})$ which is just the negative value of the slope of the line passing through E and H. Thus the line that bisects $\angle EHG$ is a vertical line passing through H, and since the x-coordinates of H and C are the same, this line must pass through C.
5. The line passing through E and H is given by the equation
$y = -y_c + (\frac{\frac{1}{2}(y_c+\sqrt{3}(x_c-1))+y_c}{\frac{1}{2}(x_c-y_c\sqrt{3}+1)-x_c})(x - x_c)$. We then verify that D lies on this line since
$-y_c + (\frac{\frac{1}{2}(y_c+\sqrt{3}(x_c-1))+y_c}{\frac{1}{2}(x_c-y_c\sqrt{3}+1)-x_c})(x_d - x_c) - y_d$
$= -y_c + (\frac{\frac{1}{2}(y_c+\sqrt{3}(x_c-1))+y_c}{\frac{1}{2}(x_c-y_c\sqrt{3}+1)-x_c})(\frac{1}{2}(x_c - y_c\sqrt{3}) - x_c) - \frac{1}{2}(x_c\sqrt{3} + y_c)$
$= 0$ as desired.
6. The line passing through G and H is given by the equation
$y = -y_c + (\frac{\frac{1}{2}(-x_c\sqrt{3}+y_c)+y_c}{\frac{1}{2}(x_c+y_c\sqrt{3})-x_c})(x - x_c)$.
We then verify that F lies on this line since
$-y_c + (\frac{\frac{1}{2}(-x_c\sqrt{3}+y_c)+y_c}{\frac{1}{2}(x_c+y_c\sqrt{3})-x_c})(x_f - x_c) - y_f$
$= -y_c + (\frac{\frac{1}{2}(-x_c\sqrt{3}+y_c)+y_c}{\frac{1}{2}(x_c+y_c\sqrt{3})-x_c})(\frac{1}{2}(x_c + y_c\sqrt{3} + 1) - x_c) - \frac{1}{2}(y_c + \sqrt{3}(1 - x_c))$
$= 0$ as desired.
7. The line passing through I and L is given by the equation

$y = -\frac{\sqrt{3}}{2} + (\frac{-\frac{\sqrt{3}}{2}-(\frac{2x_cy_c}{x_c^2+y_c^2})}{\frac{1}{2}-(\frac{x_c^2-y_c^2}{x_c^2+y_c^2})})(x-\frac{1}{2})$.

We then verify that E lies on this line since

$-\frac{\sqrt{3}}{2} + (\frac{-\frac{\sqrt{3}}{2}-(\frac{2x_cy_c}{x_c^2+y_c^2})}{\frac{1}{2}-(\frac{x_c^2-y_c^2}{x_c^2+y_c^2})})(x_e-\frac{1}{2}) - y_e$

$= -\frac{\sqrt{3}}{2} + (\frac{-\frac{\sqrt{3}}{2}-(\frac{2x_cy_c}{x_c^2+y_c^2})}{\frac{1}{2}-(\frac{x_c^2-y_c^2}{x_c^2+y_c^2})})(\frac{1}{2}(x_c - y_c\sqrt{3}+1)-\frac{1}{2}) - \frac{1}{2}(y_c+\sqrt{3}(x_c-1))$

$= 0$ as desired.

8. The line passing through J and K is given by the equation

$y = \frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1} + (\frac{\frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1}-\frac{\sqrt{3}}{2}}{\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1}-\frac{1}{2}})(x-(\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1}))$.

We then verify that D lies on this line since

$\frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1} + (\frac{\frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1}-\frac{\sqrt{3}}{2}}{\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1}-\frac{1}{2}})(x_d-(\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1})) - y_d$

$= \frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1} + (\frac{\frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1}-\frac{\sqrt{3}}{2}}{\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1}-\frac{1}{2}})(\frac{1}{2}(x_c-y_c\sqrt{3})-(\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1})) - (\frac{1}{2}(x_c\sqrt{3}+y_c))$

$= 0$ as desired.

9. The line passing through I and J is given by the equation

$y = -\frac{\sqrt{3}}{2} + (\frac{-\frac{\sqrt{3}}{2}-(\frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1})}{\frac{1}{2}-(\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1})})(x-\frac{1}{2})$.

We then verify that G lies on this line since

$-\frac{\sqrt{3}}{2} + (\frac{-\frac{\sqrt{3}}{2}-(\frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1})}{\frac{1}{2}-(\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1})})(x_g-\frac{1}{2}) - y_g$

$= -\frac{\sqrt{3}}{2} + (\frac{-\frac{\sqrt{3}}{2}-(\frac{2y_c(1-x_c)}{x_c^2-2x_c+y_c^2+1})}{\frac{1}{2}-(\frac{2y_c^2}{x_c^2-2x_c+y_c^2+1})})(\frac{1}{2}(x_c+y_c\sqrt{3})-\frac{1}{2}) - \frac{1}{2}(-x_c\sqrt{3}+y_c)$

$= 0$ as desired.

10. The line passing through K and L is given by the equation

$y = \frac{\sqrt{3}}{2} + (\frac{\frac{\sqrt{3}}{2}-(\frac{2x_cy_c}{x_c^2+y_c^2})}{\frac{1}{2}-(\frac{x_c^2-y_c^2}{x_c^2+y_c^2})})(x-\frac{1}{2})$.

We then verify that F lies on this line since

$\frac{\sqrt{3}}{2} + (\frac{\frac{\sqrt{3}}{2}-(\frac{2x_cy_c}{x_c^2+y_c^2})}{\frac{1}{2}-(\frac{x_c^2-y_c^2}{x_c^2+y_c^2})})(x_f-\frac{1}{2}) - y_f$

$= \frac{\sqrt{3}}{2} + (\frac{\frac{\sqrt{3}}{2}-(\frac{2x_cy_c}{x_c^2+y_c^2})}{\frac{1}{2}-(\frac{x_c^2-y_c^2}{x_c^2+y_c^2})})(\frac{1}{2}(x_c+y_c\sqrt{3}+1)-\frac{1}{2}) - \frac{1}{2}(y_c+\sqrt{3}(1-x_c))$

$= 0$ as desired. □

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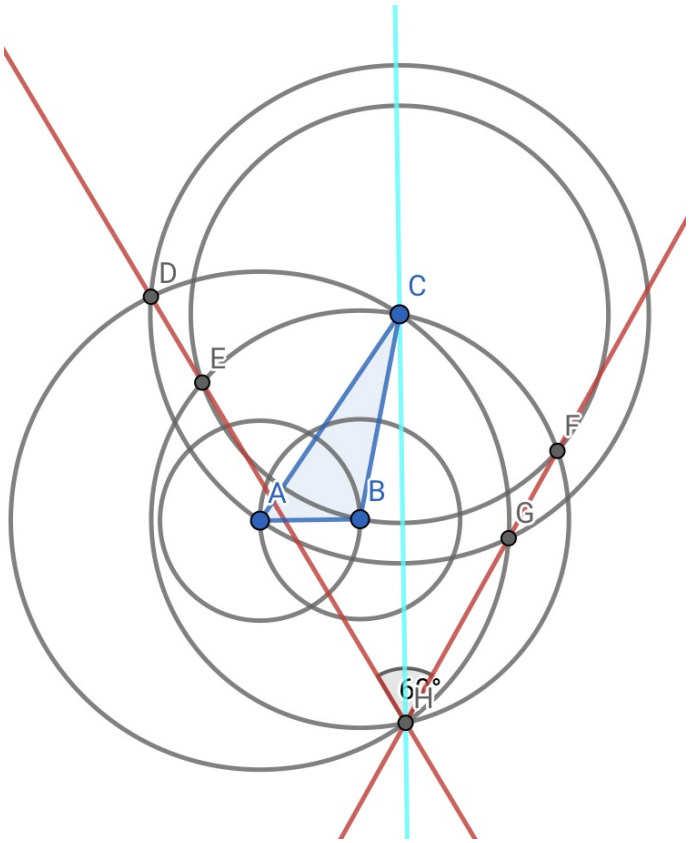

Figure 1. D, E, and H are collinear. F, G, and H are collinear. $\angle EHG = \frac{\pi}{3}$. The line that bisects $\angle EHG$ passes through C.

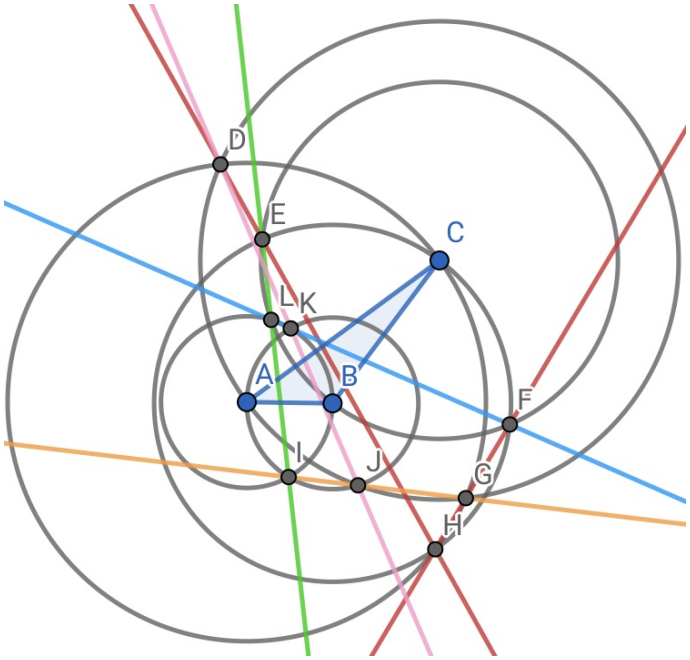

Figure 2. $|AB||GH| = |AC||KL|$ and $|AB||FH| = |BC||IJ|$. The triples of points $(E, I, L), (D, J, K)$, (G, I, J), and (F, K, L) are each collinear.