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# A Property of Certain Triangles Involving the Nine Point Circle

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## 1. Introduction

For a given triangle, consider the three circles which are each centered at a vertex and pass through the midpoint of the respective opposite side of that vertex. In late 2003, Andrew Crane [1] noticed that if these three circles intersect at a point, then that point lies on the Euler line. More specifically, he noticed that the power center of these circles (which is X(546) in Kimberling's Encyclopedia of Triangle Centers) lies on the Euler line.
Here, we consider triangles where these three circles are concurrent at a point, and we prove that this point lies on the nine point circle and that the nine point center lies on the circumcircle.

## 2. Preliminaries

Let $A$, $B$, and $C$ be the vertices of a given triangle.
Let $D$ denote the midpoint of $A$ and $B$.
Let $E$ denote the midpoint of $A$ and $C$.
Let $F$ denote the midpoint of $B$ and $C$.
Let $\mathcal{O}A$ denote the circle centered at $A$ which passes through $F$.
Let $\mathcal{O}B$ denote the circle centered at $B$ which passes through $E$.
Let $\mathcal{O}C$ denote the circle centered at $C$ which passes through $D$.

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## 3. Results

**Theorem 3.1.** *Suppose that $\mathcal{O}A$, $\mathcal{O}B$, and $\mathcal{O}C$ are concurrent at point $G$. Then $G$ lies on the nine point circle.*

*Proof.* Without loss of generality, let the cartesian coordinates of the vertices $A$, $B$, and $C$ be given respectively by $(0,0),(1,0)$, and $(x_c,y_c)$ where $x_c > 1$ and $y_c > 0$.
$D$ has coordinates $(\frac{1}{2},0)$, $E$ has coordinates $(\frac{1}{2}x_c,\frac{1}{2}y_c)$, and $F$ has coordinates $(\frac{1}{2}(x_c+1),\frac{1}{2}y_c)$.
$\mathcal{O}A$ is given by the equation $x^2+y^2=\frac{1}{4}(x_c+1)^2+\frac{1}{4}y_c^2$.
$\mathcal{O}B$ is given by the equation $(x-1)^2+y^2=(\frac{1}{2}x_c-1)^2+\frac{1}{4}y_c^2$.
Assume that $\mathcal{O}A$ and $\mathcal{O}B$ intersect at two different points (denote the point with the smaller $y$-coordinate as $G$). Solving for $x_G$ and $y_G$ gives
$x_G=\frac{1}{8}+\frac{3}{4}x_c$ and $y_G=-\frac{1}{8}\sqrt{-20x_c^2+20x_c+16y_c^2+15}$.
Assume that $\mathcal{O}A,\mathcal{O}B$, and $\mathcal{O}C$ are concurrent at $G$. This forces $x_c$ and $y_c$ to obey $(x_G-x_c)^2+(y_G-y_c)^2=(\frac{1}{2}-x_c)^2+y_c^2$.
Solving for $y_c$ now gives
$y_c=\frac{1}{6}\sqrt{12x_c^2-12x_c+6\sqrt{64x_c^4-128x_c^3+52x_c^2+12x_c+9}-18}$.
Now, the nine point circle (the circle passing through $D,E$, and $F$) is given by the equation
$(x-\frac{1}{2}x_c-\frac{1}{4})^2+(y-\frac{y_c^2-x_c^2+x_c}{4y_c})^2=(\frac{1}{4}-\frac{1}{2}x_c)^2+(\frac{y_c^2-x_c^2+x_c}{4y_c})^2$.
To verify that $G$ indeed lies on this circle, we solve for $y$ in the equation
$(x_G-\frac{1}{2}x_c-\frac{1}{4})^2+(y-\frac{y_c^2-x_c^2+x_c}{4y_c})^2=(\frac{1}{4}-\frac{1}{2}x_c)^2+(\frac{y_c^2-x_c^2+x_c}{4y_c})^2$ and we get
$y=-\frac{x_c^2}{4y_c}+\frac{x_c}{4y_c}+\frac{y_c}{4}-\frac{\sqrt{3y_c^2(2x_c-1)^2+4(-x_c^2+x_c+y_c^2)^2}}{8y_c}$.
But if we make $y=y_G$ here and set
$-\frac{x_c^2}{4y_c}+\frac{x_c}{4y_c}+\frac{y_c}{4}-\frac{\sqrt{3y_c^2(2x_c-1)^2+4(-x_c^2+x_c+y_c^2)^2}}{8y_c}=-\frac{1}{8}\sqrt{-20x_c^2+20x_c+16y_c^2+15}$
Then it follows that we must have
$y_c=\frac{1}{6}\sqrt{12x_c^2-12x_c+6\sqrt{64x_c^4-128x_c^3+52x_c^2+12x_c+9}-18}$ which is what we independently calculated above as desired. □

**Theorem 3.2.** *Suppose that $\mathcal{O}A$, $\mathcal{O}B$, and $\mathcal{O}C$ are concurrent at point $G$. Then the nine point center lies on the circumcircle.*

*Proof.* The nine point center has coordinates $(\frac{1}{2}x_c+\frac{1}{4},\frac{y_c^2-x_c^2+x_c}{4y_c})$.
The circumcircle is given by the equation
$(x-\frac{1}{2})^2+(y-\frac{y_c^2+x_c(x_c-1)}{2y_c})^2=\frac{1}{4}+(\frac{y_c^2+x_c(x_c-1)}{2y_c})^2$.
Now we solve for $y_c$ in the equation
$(\frac{1}{2}x_c+\frac{1}{4}-\frac{1}{2})^2+(\frac{y_c^2-x_c^2+x_c}{4y_c}-\frac{y_c^2+x_c(x_c-1)}{2y_c})^2=\frac{1}{4}+(\frac{y_c^2+x_c(x_c-1)}{2y_c})^2$
and so we get $y_c=\frac{1}{6}\sqrt{12x_c^2-12x_c+6\sqrt{64x_c^4-128x_c^3+52x_c^2+12x_c+9}-18}$ which is what we independently calculated above as desired. □

## References

Kimberling, C. X(546), *The Encyclopedia of Triangle Centers*, (https://faculty.evansville.edu/ck6/encyclopedia/ETC.htmlX546)

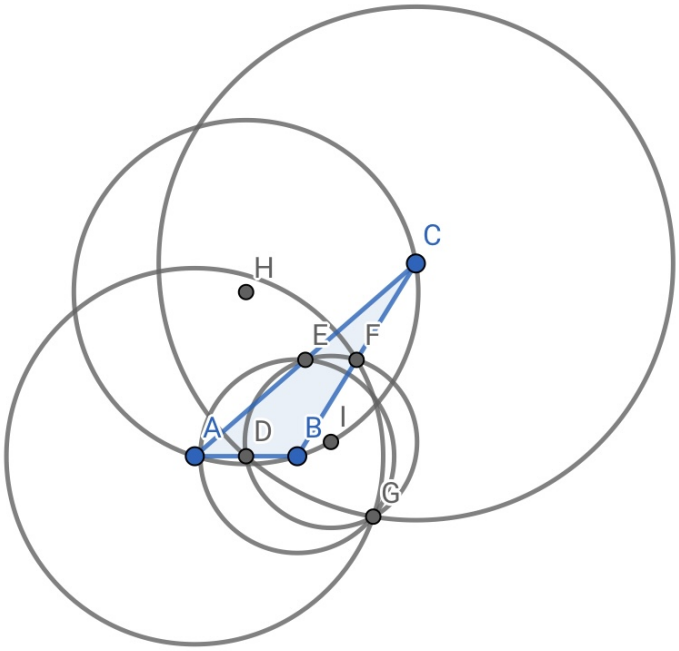


Figure 1. If $\mathcal{O}A$, $\mathcal{O}B$, and $\mathcal{O}C$ all intersect at $G$, then $G$ lies on the nine point circle (the circle passing through $D, E$, and $F$), and the nine point center $I$ lies on the circumcircle.